PUBLISHED BY INSTITUTE OF PHYSICS PUBLISHING FOR SISSA/ISAS
RECEIVED: *January 27, 2004*
ACCEPTED: *February 11, 2004*

Representations of $\mathcal{G}^{+++}$ and the role of space-time

Axel Kleinschmidt
Department of Applied Mathematics and Theoretical Physics
Centre for Mathematical Sciences
Wilberforce Road, Cambridge CB3 0WA, U.K.
E-mail: a.kleinschmidt@damtp.cam.ac.uk

Peter West
Department of Mathematics
King's College, London WC2R 2LS, U.K.
E-mail: pwest@mth.kcl.ac.uk

ABSTRACT: We consider the decomposition of the adjoint and fundamental representations of very extended Kac-Moody algebras $\mathcal{G}^{+++}$ with respect to their regular A type subalgebra which, in the corresponding non-linear realisation, is associated with gravity. We find that for many very extended algebras almost all the A type representations that occur in the decomposition of the fundamental representations also occur in the adjoint representation of $\mathcal{G}^{+++}$. In particular, for E_8^{+++}, this applies to all its fundamental representations. However, there are some important examples, such as A_{N-3}^{+++}, where this is not true and indeed the adjoint representation contains no generator that can be identified with a space-time translation. We comment on the significance of these results for how space-time can occur in the non-linear realisation based on $\mathcal{G}^{+++}$. Finally we show that there is a correspondence between the A representations that occur in the fundamental representation associated with the very extended node and the adjoint representation of $\mathcal{G}^{+++}$ which is consistent with the interpretation of the former as charges associated with brane solutions.

KEYWORDS: M-Theory, String Duality, Space-Time Symmetries, Gauge Symmetry.

JHEP02(2004)033

Contents

1. Introduction

It is a consequence of supersymmetry that the scalars in supergravity multiplets belong to non-linear realisations [1]. One of the most celebrated examples concerns the four dimensional maximal supergravity where the scalars belong to a non-linear realisation of E_7 [2]. The eleven dimensional supergravity theory does not possess any scalars and it was widely believed that these symmetry algebras were not present in this theory. However, it was found that the eleven dimensional supergravity theory could be formulated as a non-linear realisation [3]. The infinite dimensional algebra involved in this construction was the closure of a finite dimensional algebra, denoted G_{11}, with the eleven dimensional conformal algebra. The non-linear realisation was carried out by ensuring that the equations of motion were invariant under both finite dimensional algebras, taking into account that some of their generators were in common. The algebra G_{11} involved the space-time translations together with an algebra $\hat{G}_{11}$ which contained A_{10} and the Borel subalgebra of E_7 as subalgebras.

The algebra $\hat{G}_{11}$ was not a Kac-Moody algebra, however, it was conjectured [4] that the theory could be extended so that the algebra $\hat{G}_{11}$ was promoted to a Kac-Moody algebra. It was shown that this symmetry would have to contain a certain rank eleven Kac-Moody algebra denoted E_{11} [4].

Consequently, it was argued [4] that an extension of eleven dimensional supergravity should possess an E_{11} symmetry that was non-linearly realised. In particular, the symmetries found when the eleven dimensional supergravity theory was dimensionally reduced would be present in this eleven dimensional theory. One of the advantages of a non-linear realisation is that the dynamics is largely specified by the algebra if the chosen local subalgebra is sufficiently large. This was not the case for the G_{11} considered in [3] when taken in isolation as the local subalgebra was choosen to be just the Lorentz algebra, but it is the case for E_{11} with the local subalgebra that was specified in [4]. However, the way the space-time generators and the conformal algebra should relate to the E_{11} non-linear realisation was not considered in reference [4].

A similar picture emerged for the IIA and IIB supergravity theories in ten dimensions and it was conjectured that these theories could be extended such that they were invariant under E_{11} [4, 5]. As a result, the different type-II theories arose by taking different local subalgebras of E_{11} [4, 5].

Similar ideas to that advocated in [4] were adopted in [6] where it was argued that eleven dimensional supergravity was invariant under a non-linear realisation of the E_{10} subalgebra of E_{11}. However, guided by earlier work on cosmological billiards [7], this work treated space-time in a way that was different to that considered in [3], but consistent with the conjecture of [4]. The fields in the non-linear realisation were presumed to depend only on time and any spatial dependence was to arise in the equations of motion resulting from the non-linear realisation. At the level at which this construction was carried out, the fields were almost independent of spatial coordinates, but it was observed that the E_{10} adjoint representation contained fields with the correct A_9 representations to be the spatial derivatives of some of the fields that occcur in eleven dimensional supergravity. Put another way, it was supposed that the spatial dependence would arise from the structure of E_{10}. However, the space-time translation operator was not thought to appear as one of the generators of E_{11}. For the non-linear realisation of a Kac-Moody algebra with a local subalgebra which is chosen to be that which is invariant under the Cartan involution there exists an invariant lagrangian [8]. Hence, regarding the spatial dependence to arise in E_{10} itself had the additional technical advantage that one could use this lagrangian formulation.

An alternative proposal to incorporate the space-time generators into the context of a non-linear realisation of E_{11} was made in reference [9]. The space-time translations naturally occur as the lowest level object in the fundamental representation associated with the very extended node, denoted l_1, of E_{11}. It was proposed to take the non-linear realisation of the algebra formed from the E_{11} semi-direct product of E_{11} with generators corresponding to the l_1 representation. This had the advantage that at low levels this construction was that carried out in reference [3] and so one did obtain a theory that described correctly eleven dimensional supergravity in terms of fields that had a full

JHEP02(2004)033

space-time dependence. One not anticipated result in this approach was that the l_1 representation contains at its first few levels the space-time translations as well as the central charges found in the supersymmetry algebra together with an infinite number of other states.

Very recently an alternative approach was advocated [10]; as in [4] the theory was assumed to be a non-linear realisation of E_{11}, but the fields were assumed to depend on an auxiliary parameter and, in the spirit of [6] all the space-time dependence was assumed to arise from the equations of motion of the non-linear realisation in some way. It was also noted, by examing the tables of [11], that at level seven a generator occurred with the correct A_{10} representation to be identified as a space-time generator and it was supposed that this might provide a mechanism for introducing space-time into the fields.

A similar scenario to that advocated to eleven dimensional supergravity in [4] was supposed to occur for gravity [12] in D dimensions and the effective action of the closed bosonic string [4] generalised to D dimensions. It was realised that the algebras that arose in all these theories were of a special kind and were called very extended Kac-Moody algebras [13]. Indeed, for any finite dimensional semi-simple Lie algebra $\mathcal{G}$ one can systematically extend its Dynkin diagram by adding three more nodes to obtain an indefinite Kac-Moody algebra denoted $\mathcal{G}^{+++}$. The algebras for gravity and the closed bosonic string being D_{D-2}^{+++} [12] and A_{D-3}^{+++} [5] respectively. This general approach was extended in reference [14] where the non-linear realisation of $\mathcal{G}^{+++}$ restricted to its Cartan subalgebra was constructed and the Weyl transformations resulting were shown to transform the Kasner solutions into each other and, for $E_8^{+++} = E_{11}$ and D_{24}^{+++}, were shown to be related to U duality transformations in the corresponding string theories. In [15] it was shown that the theories associated with $\mathcal{G}^{+++}$ admit BPS intersecting solutions. Finally, it was shown [16] that the low level content of the adjoint representation of $\mathcal{G}^{+++}$ predicted a field content for a non-linear realisation of $\mathcal{G}^{+++}$ consistent with oxidation theory. As a result of [4, 12, 13] and [14, 15, 16, 10], one was lead to suppose, as was explicitly stated in the latter papers, that the non-linear realisation of any very extended algebra $\mathcal{G}^{+++}$ leads to a theory, called $\mathcal{V}_{\mathcal{G}}$ in [16], that at low levels includes gravity and the other fields found in the corresponding oxidised theory and it was hoped that this non-linear realisation contains an infinite number of propagating fields that ensures the consistency of these theories.

As the above explains, there is growing evidence for the conjecture set out in [4], namely that an extension of eleven dimensional supergravity possessing an E_{11} (or a subalgebra of this) symmetry which is non-linearly realised. However, there are different proposals as to how space-time should arise in this construction. To help resolve these differences in this paper we investigate the content of the fundamental and adjoint representations of $\mathcal{G}^{+++}$. In particular we will, following [17, 16], analyse $\mathcal{G}^{+++}$ and its representations in terms of the most physically relevant A subalgebra which in the non-linear realisation is associated with gravity and find relations between the content of the fundamental representations and adjoint representation of $\mathcal{G}^{+++}$. We will show how the analogues of the representation proposed for E_8^{+++} in [9] contain precisely the right fields at lowest levels to be interpreted as central charges. In addition, we analyse the different proposals for how spacetime enters in the light of our findings.

JHEP02(2004)033

The structure of this paper is as follows. Section 2 contains the decomposition strategy for $\mathcal{G}^{+++}$ and its representations and the results obtained by this method. In section 3 we show how the so-called l_1 representation contains the correct fields in correspondence to the generalisations of the central charges of supersymmetry algebras. Section 4 discusses the consequences of the results in the earlier sections for the role of space-time. Appendix A lists all the Dynkin diagrams of the very extended algebras, appendix B tabulates a number of $\bar{l}_1$ representations of various $\mathcal{G}^{+++}$ analysed to a greater height. In appendix C we also list the $\bar{l}_1$ decomposition for $E_{10} = E_8^{++}$.

2. Representations and analysis of indefinite Kac-Moody algebras

2.1 Relationship between fundamental and adjoint representations of $\mathcal{G}^{+++}$

Given a semi-simple finite dimensional Lie algebra $\mathcal{G}$ we can construct its very extension $\mathcal{G}^{+++}$ [13]. We label the Dynkin diagram of $\mathcal{G}^{+++}$ by assigning to the very, over and affine nodes the labels 1,2,3 and continue this labelling along the horizontal line of the Dynkin diagram going from left to right and then label the nodes above the horizontal working from right to left. The Dynkin diagrams with this labelling are given in appendix A. We are interested in the fundamental representations of $\mathcal{G}^{+++}$. These are the standard highest weight representations of $\mathcal{G}^{+++}$ with highest weights given by the fundamental weights of $\mathcal{G}^{+++}$. We will study this problem by considering the fundamental representations embedded in an extended algebra denoted $\mathcal{G}^{+++a}$ whose Dynkin diagram is obtained from that of $\mathcal{G}^{+++}$ by adding one more node attached by a single line to the node labelled a of $\mathcal{G}^{+++}$. We label this additional node by 0. The representations of Kac-Moody algebras could also be calculated from the Weyl-Kac character formula or the Freudenthal formula [18] but using this auxiliary extended diagram is more suited to our analysis of the relations between the algebra and its representations. In order to reveal parts of the structure of the infinite dimensional $\mathcal{G}^{+++}$ we consider a decomposition of $\mathcal{G}^{+++}$ with respect to a finite dimensional subalgebra as in reference [13]. There is (at least) one node c of the Dynkin diagram of $\mathcal{G}^{+++}$ whose deletion leaves the diagram of a finite dimensional semi-simple Lie algebra. We denote this subalgebra by $\mathcal{H}$. For example, for the case of E_8^{+++}, this can be the node labelled 11 and the algebra remaining after its deletion is A_{10}.

The roots of $\mathcal{G}^{+++a}$ can be written in terms of simple roots in the form

$$\beta = n_0\alpha_0 + n_c\alpha_c + \sum_i n_i\alpha_i \,. \tag{2.1}$$

Our notation here is that Latin indices from the middle of the alphabet run over the subalgebra $\mathcal{H}$, whereas letters from the beginning run over $\mathcal{G}^{+++}$. The concept of level was implicit in reference [13], given explicitly for E_{10} in reference [6] and worked out in general in [17, 16]. In our case, each root β has two levels, (n_c, n_0), which are the numbers of times the simple roots α_c and α_0 occur in the root expansion. The roots of level $(\star, 0)$, where $\star$ is any level, are those of $\mathcal{G}^{+++}$ and amongst these the roots of level $(0,0)$ generate the finite dimensional subalgebra $\mathcal{H}$ in terms of which we will analyse the algebra $\mathcal{G}^{+++a}$. (There are also additional scalars from the Cartan subalgebra generators of the deleted nodes.) As the notion of level is preserved by the commutators, the roots of level $(\star, 1)$ form a representation of $\mathcal{G}^{+++}$. It is just the standard lowest weight representation with lowest weight $\bar{l}_a$ of $\mathcal{G}^{+++}$, which we will refer to as the fundamental $\bar{l}_a$ representation. The fundamental weights of $\mathcal{G}^{+++}$ are defined as usual by

$$\frac{2(\alpha_a, l_b)}{(\alpha_a, \alpha_a)} = \delta_{ab} \,. \tag{2.2}$$

Here the inner product is taken in the weight lattice of $\mathcal{G}^{+++}$. The fundamental weights give rise to highest weight representations with highest weight l_a. For the lowest weight representations we are considering we have $\bar{l}_a = -l_a$. For computational purposes it is more convenient to consider the lowest weight representations $\bar{l}_a$ and their $\mathcal{H}$ representation content is identical to that of the highest weight representation l_a. We also note that $(\star, -1)$ also forms a representation, namely just the highest weight l_a representation. We first consider decomposing the algebra $\mathcal{G}^{+++a}$ by deleting the node 0. Following [13] we can write the simple roots as

$$\alpha_0 = y - l_a \,, \qquad \alpha_c \,, \qquad \alpha_i \tag{2.3}$$

where α_i, α_c are the simple roots of G^{+++} and y is orthogonal to these in the inner product of the extended algebra $\mathcal{G}^{+++a}$ as can be seen by taking the inner product with any simple α_b of $\mathcal{G}^{+++}$ and using the definition of the fundamental weights. As we will be dealing with a number of consecutive subalgebras of $\mathcal{G}^{+++a}$, we have to fix an embedding of the different weight lattices in the weight lattice of $\mathcal{G}^{+++a}$. We do this horizontally, i.e. by filling in zeros in the simple root expansion of any element for those components which require embedding. Using this convention we do not need to put labels on the different inner products to distinguish them and it should be clear from the context which space we are working in. From (2.3) we see that $y^2 + l_a^2 = 2$ since $\alpha_0^2 = 2$.

We can further decompose the algebra by deleting the node labelled c. Here we assume for simplicity that a is not attached to c, but an analogous construction works in this case, for example using the formalism of [13, 16]. Taking into account the possiblility of starting from non-simply laced algebras $\mathcal{G}$, we may write the simple roots of $\mathcal{G}^{+++}$ as

$$\alpha_c = x - \nu \,, \qquad \alpha_i \tag{2.4}$$

where α_i are the roots of H and

$$\nu = -\sum_j A^{\mathcal{G}^{+++}}_{cj} \lambda_j \frac{(\alpha_c, \alpha_c)}{(\alpha_j, \alpha_j)} \tag{2.5}$$

where λ_j are the fundamental weights of $\mathcal{H}$. The fundamental weights of $\mathcal{G}^{+++}$ are given by

$$l_i = \lambda_i + \frac{x}{x^2}(\nu, \lambda_i) \,, \qquad l_c = \frac{x}{x^2} \,. \tag{2.6}$$

We note that $x^2 + \nu^2 = \alpha_c^2$. One can show for the simply laced case that $\det A^{\mathcal{G}^{+++}} = -2 \det A^{\mathcal{G}} = x^2 \det A^{\mathcal{H}}$ [13]. Here and throughout the paper $A^{\mathcal{X}}$ denotes the generalised Cartan matrix of the Kac-Moody algebra $\mathcal{X}$, or without superscript it will refer to the preferred A type subalgebra.

JHEP02(2004)033

As a result of the above decompositions we can write the roots of $\mathcal{G}^{+++a}$ given in equation (2.1) as

$$\beta = n_0 y + x\left(n_c - n_0\frac{(\nu,\lambda_a)}{x^2}\right) - \Lambda \tag{2.7}$$

where

$$\Lambda = n_0\lambda_a + n_c\nu - \sum_i n_i\alpha_i\,. \tag{2.8}$$

We note that Λ belongs to the weight lattice of $\mathcal{H}$. As explained in [6, 17, 16], if the adjoint representation of $\mathcal{G}^{+++a}$ contains generators in the representation of $\mathcal{H}$ with highest weight $\sum_j p_j\lambda_j$, p_j being the Dynkin labels, then the following condition must hold

$$\sum_j p_j\lambda_j = \Lambda = n_0\lambda_a + n_c\nu - \sum_i n_i\alpha_i \tag{2.9}$$

where p_j, n_0, n_c and n_i are all non-negative integers. Taking the scalar product with λ_k this condition becomes

$$\sum_j p_j(\lambda_j,\lambda_k) = n_0(\lambda_a,\lambda_k) + n_c(\nu,\lambda_k) - n_k\,. \tag{2.10}$$

In principle, there should be a factor of $\frac{2}{(\alpha_k,\alpha_k)}$ with the term n_k but in the cases of interest to us, the subalgebra will be of simply laced type and so this factor is one. We may invert this equation for a general Kac-Moody algebra $\mathcal{X}$ in terms of the inverse Cartan matrix by using the relation

$$(A^{\mathcal{X}})^{-1}_{ab} = \frac{2}{(\alpha_a,\alpha_a)}(\lambda_i,\lambda_b)\,. \tag{2.11}$$

The scalar product of the roots of $\mathcal{G}^{+++a}$ take the form

$$\beta^2 = n_0^2y^2 + x^2\left(n_c - n_0\frac{(\nu,\lambda_a)}{x^2}\right)^2 + \sum_{ij} p_i(\lambda_i,\lambda_j)p_j = 2,1,0,\ldots\,. \tag{2.12}$$

In the second equation we have inserted the constraint that the roots of a Kac-Moody algebra must have lengths squared which are integers 2 or less [18], except for G_2^{+++} where short roots have length squared 2/3. For a Kac-Moody algebra with symmetric Cartan matrix we can get only even integers.

Equations (2.10) and (2.12) are necessary conditions for the representation of $\mathcal{G}$ with Dynkin indices p_j to belong to the adjoint representation of $\mathcal{G}^{+++a}$. However, they are not as strong as implementing all the Serre relations and it can happen that they possess some solutions that do not actually occur in the adjoint representation. In fact, this possibility does occur, but it seems rather exceptional, as the reader can verify by looking at the tables in appendix B and C, see also comments in [11, 17, 16]. All the statements made below are assumed to be modulo this exception. We also note that for each fixed level (n_c, n_0) there are only a finite number of possible solutions to this equation. (In the tables we have also imposed the additional constraint that β is an actual root of $\mathcal{G}^{+++a}$ with non-vanishing multiplicity.) The outer multiplicity μ with which a solution to (2.10) and (2.12) appears can be computed recursively using knowledge about the multiplicities of the roots of $\mathcal{G}^{+++a}$, obtained from computations based on the denominator and Peterson formula [18].

Before beginning the detailed analysis, we make some general remarks. Given a finite dimensional semi-simple Lie algebra, its adjoint representation is a highest weight representation and it is contained in the tensor products of the fundamental representations. Also, any highest weight representation of a finite dimensional algebra also has a lowest weight. However, for Kac-Moody algebras things are very different. The adjoint representation is not a highest weight representation (nor a lowest weight representation) and it is not contained in the tensor product of the fundamental representations. This is easily seen by considering the decomposition of a Kac-Moody algebra. For simplicity, let us consider a lorentzian Kac-Moody algebra such that one can delete one node from its Dynkin diagram to find a finite dimensional semi-simple Lie algebra [13]. The algebra possesses a root decomposition that contains roots at all positive and negative levels and so there is no highest weight state. A fundamental representation has the decomposition of equation (2.6). A crucial factor in the analysis of the representation depends on the sign of x^2 which is positive for finite dimensional semi-simple Lie algebras and negative for very extended Kac-Moody algebras . As a result, the highest weight state has a positive level for a finite dimensional semi-simple Lie algebra and a negative level for a very extended Kac-Moody algebra. Therefore, what we are considering for positive n_0 are lowest weight representations. The root string of the fundamental highest weight representation l_a contains a weight of the form $l_a - n_c\alpha_c - \sum_i n_i\alpha_i$ and so the level of the weights in the representation is always less than that of the highest weight states. As such, for the Kac-Moody case, the states in the fundamental highest weight representation have only negative levels and so by taking their tensor product one can never obtain the positive level states found in the adjoint representation. One way that one might find the adjoint representation is to take tensor products of the highest and lowest weight representations. Indeed, we have checked, admittedly at rather low levels, that the A_9 representations found in the adjoint representation of E_8^{++} are contained in the tensor product of the l_1 and the l_1^* representation at the appropriate level.

In contrast, for finite dimensional semi-simple Lie algebras, the fundamental representations have a positive level and, as the level decreases as one contructs the root string, one finds in the representation states of positive and negative levels consistent with the fact that tensor products of these representations contain the adjoint representation.

Now we return to our discussion of the representations contained in $\mathcal{G}^{+++}$ and its fundamental representations $\bar{l}_a$. The roots in the adjoint representation of $\mathcal{G}^{+++}$ are just those for which $n_0 = 0$ and so they will possess lowest weight representation of $\mathcal{H}$, with Dynkin indices $p_j^{(0)}$, if

$$\sum_j p_j^{(0)}(\lambda_j,\lambda_k) = n_c^{(0)}(\nu,\lambda_k) - n_k^{(0)} \,. \tag{2.13}$$

The $\bar{l}_a$ representation of $\mathcal{G}^{+++}$ is given by just those elements in the adjoint representation of $\mathcal{G}^{+++a}$ with $n_0 = 1$. Therefore they will contain the highest weight representation of $\mathcal{H}$, with Dynkin indices $p_j^{(1)}$, if

$$\sum_j p_j^{(1)}(\lambda_j,\lambda_k) = (\lambda_a,\lambda_k) + n_c^{(1)}(\nu,\lambda_k) - n_k^{(1)} \,. \tag{2.14}$$

We note that the lowest $\mathcal{H}$ representation in the fundamental $\bar{l}_a$ representation occurs at level $n_c = 0$ and has Dynkin index $p_a^{(1)} = 1$, all the other Dynkin indices being zero. It corresponds to the root of $\mathcal{G}^{+++a}$ with components $(1, 0, \ldots, 0)$.

Let us consider a representation of $\mathcal{H}$ with Dynkin indices $p_j^{(1)}$ that occurs in the fundamental representation $\bar{l}_a$ of $\mathcal{G}^{+++}$, subtracting the two above equations, we find that this $\mathcal{H}$ representation will also occur in the adjoint representation of $\mathcal{G}^{+++}$ provided that the equation

$$(\lambda_a + (n_c^{(1)} - n_c^{(0)})\nu, \lambda_k) = n_k^{(1)} - n_k^{(0)} \tag{2.15}$$

holds for positive $n_k^{(0)}$. Here $n_c^{(1)} - n_c^{(0)}$ can be interpreted as a shift in level. The converse result holds if $n_k^{(1)}$ is positive. As a result, we conclude that $(\lambda_a + (n_c^{(1)} - n_c^{(0)})\nu)$ is in the dual of the weight lattice $(\Lambda^{\mathcal{H}})^*$ and so is in the orignal root lattice of $\mathcal{H}$, $\Lambda^{\mathcal{H}}$. As we noted above the the lowest H representation in the fundamental representation $\bar{l}_a$ has Dynkin indices $p_j^{(0)} = \delta_{aj}$. Using equation (2.13), the condition for such a representation to occur in the adjoint representation at level n_c is $(\lambda_a - n_c\nu, \lambda_k) = -n_k$. This in agreement with the condition of equation (2.15), but it also tells us that if we find the lowest component of the $\bar{l}_a$ representation in the adjoint we also may find all the higher $\bar{l}_a$ states in the adjoint representation, in a way that is consistent with the action of $\mathcal{G}^{+++}$ on the algebra and on the fundamental representation.

Let us consider E_8^{+++}, for which $\mathcal{H} = A_{10}$, and

$$\nu = \lambda_8\,, \qquad x^2 = -\frac{2}{11}\,. \tag{2.16}$$

Equation (2.15) tells us that an A_{10} representation that occurs in the fundamental representation $\bar{l}_a$ of E_8^{+++} will occur in the adjoint representation of E_8^{+++} if

$$A^{-1}_{ak} + (n_c^{(1)} - n_c^{(0)})A^{-1}_{8k} = n_k^{(1)} - n_k^{(0)}\,. \tag{2.17}$$

It is straightforward to analyse this equation and for odd a one finds the following solution to the equation

$$n_c^{(0)} - n_c^{(1)} = \frac{(11 + 3a)}{2}\,. \tag{2.18}$$

The root vectors being related for $a \leq 7$ by

$$n_k^{(0)} - n_k^{(1)} = \begin{cases} k\dfrac{(a+1)}{2}\,, & k \leq a \\ \dfrac{k}{2}(a+3) - a\,, & k \geq a\,, \quad k \leq 8 \\ (11-k)(a+4)\,, & k \geq 8 \end{cases} \tag{2.19}$$

and for $a = 9$ by

$$n_k^{(0)} - n_k^{(1)} = \begin{cases} 5k\,, & k \leq 8 \\ 26\,, & k = 9\,. \end{cases} \tag{2.20}$$

As we noted the lowest A_{10} representation in $\bar{l}_a$ has highest weight λ_a and one can read off the precise roots in $\mathcal{G}^{+++a}$ to which it corresponds. For the case of $a = 1$ we find the lowest representation has Dynkin index $p_1 = 1$, i.e. it is of the form of a momentum generator P_a, and it occurs at level $n_c^{(0)} = 7$ with a root β in $E_{12} = E_8^{+++1}$ corresponding to $(0,1,3,5,7,9,11,13,15,10,5,7)$. But there are also solutions occurring higher up in the algebra, on levels $n_c = 18, 29, \ldots$. Later we will show that there are actually solutions for all $\bar{l}_a$ in this case, which will imply that all the fundamental representations potentially sit within the adjoint.

We now consider A_{n-3}^{+++} for which the node c is the node labelled n, $\mathcal{H} = A_{n-1}$ and

$$\nu = \lambda_3 + \lambda_{n-1}\,, \qquad x^2 = -2\frac{(n-2)}{n}\,. \tag{2.21}$$

Equation (2.15) tells us that an A_{n-1} representation that occurs in the fundamental representation $\bar{l}_a$ of A_{n-3}^{+++} will occur in the adjoint representation of A_{n-3}^{+++} if

$$A_{ak}^{-1} + (n_c^{(1)} - n_c^{(0)})(A_{3k}^{-1} + A_{n-1k}^{-1}) = n_k^{(1)} - n_k^{(0)}\,. \tag{2.22}$$

Let us first consider the $\bar{l}_1$ representation then one finds that

$$n(n_k^{(1)} - n_k^{(0)}) = \begin{cases} n(1 + k(n_c^{(1)} - n_c^{(0)})) - k(1 + 2(n_c^{(1)}) - n_c^{(0)})\,, & k \leq 3 \\ n(1 + 3(n_c^{(1)} - n_c^{(0)})) - k(1 + 2(n_c^{(1)}) - n_c^{(0)})\,, & k \geq 3 \end{cases} \tag{2.23}$$

We therefore conlcude that

$$n_c^{(0)} - n_c^{(1)} = \frac{(pn+1)}{2} \tag{2.24}$$

where p is a positive integer. Clearly, if n is even this can never be the case, but for n odd we find there is always a solution by choosing $p = 1$. In fact one can use equation (2.23) to search for any representation of A_{n-1} in the adjoint of A_{n-3}^{+++} and one finds that for n even, despite the infinite number of representations present, no such representation ever occurs. Hence, it is not the case that the $\mathcal{H}$ representations in the $\bar{l}_1$ representation of $\mathcal{G}^{+++}$ always belong in the adjoint representation and indeed it can happen that the adjoint representation contains no generator that can be identified with the space-time translations.

Analysing equation (2.22) one finds that a A_{n-1} representation that occurs in the fundamental representation $\bar{l}_a$ of A_{n-3}^{+++} will occur in the adjoint representation of A_{n-3}^{+++} if

$$n_c^{(0)} - n_c^{(1)} = \frac{pn+a}{2} \tag{2.25}$$

where p is a positive integer.

Now we consider G_2^{+++}, for which the node c is the node labelled 5 and $\mathcal{H} = A_4$. We normalise the roots such that $\alpha_c^2 = 2/3$, $\alpha_4^2 = 2$ and $(\alpha_c, \alpha_4) = -1$. For this case

$$\nu = \lambda_4\,, \qquad x^2 = -\frac{2}{15}\,. \tag{2.26}$$

Equation (2.15) tells us that an A_4 representation that occurs in the fundamental lowest weight representation $\bar{l}_a$ of G_2^{+++} will occur in the adjoint representation of G_2^{+++} if

$$A_{ak}^{-1} + (n_c^{(1)} - n_c^{(0)})A_{4k}^{-1} = n_k^{(1)} - n_k^{(0)}\,. \tag{2.27}$$

Examining $\bar{l}_a$ we find there is a solution for

$$n_c^{(0)} - n_c^{(1)} = 5 - a, \; n_k^{(0)} - n_k^{(1)} = \begin{cases} 0\,, & k \le a \\ k - a\,, & k \ge a \end{cases} \tag{2.28}$$

In particular for l_1 the lowest level generator, P_a has a β in G_2^{+++1} corresponding to $(0,0,1,2,3,4)$.

Although for the case of the Dynkin diagram E_8^{+++} one can delete the node labelled 11 to obtain a sub-Dynkin diagram corresponding to A_{10}, it is not the case for all very extended algebras that one finds a subalgebra A_p for a particular p. However, in the application to non-linear realisations we have in mind there is always a preferred A type subalgebra that is associated with the gravity sector of the theory and it is the decompostion to this subalgebra that is of interest. As a result, for some very extended algebras it is desirable to delete yet one more node, labelled d, which lives on the Dynkin diagram of $\mathcal{H}$. In these cases the roots in $\mathcal{G}^{+++a}$ have three levels (n_c, n_d, n_0). For example, for the case of D_{n-3}^{+++} the node c corresponds to the node labelled n and the remaining subalgebra is D_{n-1}. We then delete the node labelled $n-1$ to find the subalgebra A_{n-2} which is the algebra that controls the gravity sub-sector of the non-linear realisation. We now find the analogue of the above constraints after this decomposition. We write the simple roots of $\mathcal{H}$ as

$$\alpha_d = x - \tau\,, \qquad \alpha_i\,, \quad i \neq d \tag{2.29}$$

where α_i are the roots of A_p and

$$\tau = -\sum_j A^H_{dj}\;\mu_j \frac{(\alpha_d, \alpha_d)}{(\alpha_j, \alpha_j)} \tag{2.30}$$

where μ_j are the fundamental weights of A_p. The fundamental weights of $\mathcal{H}$ are given by

$$\lambda_i = \mu_i + \frac{z}{z^2}(\tau, \mu_i),\; l_d = \frac{z}{z^2} \tag{2.31}$$

and so

$$\nu = -\sum_j A^{G^{+++}}_{cj}(\mu_j + \frac{z}{z^2}(\tau, \mu_a)) = \tilde{\nu} - \sum_j \frac{z}{z^2} A^{G^{+++}}_{cj}(\tau, \mu_a)\,. \tag{2.32}$$

Following the same arguments as before, we find that the adjoint representation of $\mathcal{G}^{+++}$ contains generators in the lowest weight representation of A_p, with Dynkin indices $q_j^{(0)}$, if

$$\sum_j q_j^{(0)}(\mu_j, \mu_k) = n_c^{(0)}(\tilde{\nu}, \mu_k) + n_d^{(0)}(\tau, \mu_k) - n_k^{(0)}\,. \tag{2.33}$$

On the other hand, the $\bar{l}_a$ representation of $\mathcal{G}^{+++}$ will contain generators in the lowest weight representation of A_p, with Dynkin indices $q_j^{(1)}$, if

$$\sum_j q_j^{(1)}(\mu_j, \mu_k) = (\mu_a, \mu_k) + n_c^{(1)}(\tilde{\nu}, \mu_k) + n_d^{(1)}(\tau, \mu_k) - n_k^{(1)}\,. \tag{2.34}$$

Consequently, a representation of A_p with Dynkin indices $p_j^{(1)}$ that occurs in the fundamental lowest weight representation $\bar{l}_a$ of $\mathcal{G}^{+++}$, also occurs in the adjoint representation of $\mathcal{G}^{+++}$ provided that

$$(\mu_a + (n_c^{(1)} - n_c^{(0)})\nu + (n_d^{(1)} - n_d^{(0)})\tau, \mu_k) = n_k^{(1)} - n_k^{(0)} \tag{2.35}$$

holds for $n_k^{(0)}$ positive.

Let us consider E_7^{+++}. The nodes c and d are the nodes labelled 9 and 10 respectively, the subalgebra we obtain is A_8 and

$$\tilde{\nu} = \mu_6\,, \qquad \tau = \mu_8\,. \tag{2.36}$$

Equation (2.35) becomes

$$A^{-1}_{ak} + (n_c^{(1)} - n_c^{(0)})A^{-1}_{6k} + (n_d^{(1)} - n_d^{(0)})A^{-1}_{8k} = n_k^{(1)} - n_k^{(0)}\,. \tag{2.37}$$

Considering the l_1 representation we find a soution with

$$n_c^{(0)} - n_c^{(1)} = 2\,, \qquad n_d^{(0)} - n_d^{(1)} = 2\,, \qquad n_k^{(0)} - n_k^{(1)} = \begin{cases} k-1\,, & k \leq 6\,, \\ 11-k\,, & k \geq 6\,. \end{cases} \tag{2.38}$$

The lowest level generator P_a in $\bar{l}_1$ that occurs in E_7^{+++} appears at level $(2,2)$ and has β corresponding to $(0,0,1,2,3,4,5,4,3,2,2)$.

For D_{n-3}^{+++} the nodes c and d are the nodes labelled n and $n-1$ respectively, the subalgebra we obtain is A_{n-2} and

$$\tilde{\nu} = \mu_4\,, \qquad \tau = \mu_{n-3}\,. \tag{2.39}$$

Given any A_{n-2} representation which occurs in the $\bar{l}_a$ of D_{n-3}^{+++} it will also satisfy (2.10) and (2.12) for the adjoint representation of D_{n-3}^{+++} if

$$A^{-1}_{ak} + (n_c^{(1)} - n_c^{(0)})A^{-1}_{4k} + (n_d^{(1)} - n_d^{(0)})A^{-1}_{n-3,k} = n_k^{(1)} - n_k^{(0)}\,. \tag{2.40}$$

Analysing this equation for the $\bar{l}_1$ representation we find a solution if

$$2(n_c^{(0)} - n_c^{(1)}) - (n_d^{(0)} - n_d^{(1)}) = \frac{(p(n-1)+1)}{2} \tag{2.41}$$

where p is a positive integer. Clearly, this is impossible if n is odd, but for n even we can always find a solution with $p = 1$.

Carrying out the same analysis for the $\bar{l}_a$ representation of B_{n-3}^{+++} we have to analyse

$$A^{-1}_{ak} + (n_c^{(1)} - n_c^{(0)})A^{-1}_{4k} + (n_d^{(1)} - n_d^{(0)})A^{-1}_{n-2,k} = n_k^{(1)} - n_k^{(0)}\,. \tag{2.42}$$

For the particular case of $\bar{l}_1$ we find that this equation always has a solution.

One could similarly deal with C_n^{+++} but here the process has to be repeated many times and the analysis is not very illuminating in general. One finds that there is a solution which looks like P_a in all cases.

Let us summarise the findings of this section. We have shown that the necessary conditions (2.10) and (2.12) for a representation of the subalgebra $\mathcal{H}$ of $\mathcal{G}^{+++}$ to be present in the adjoint is often equivalent to the necessary condition for the same representation to be present in the fundamental $\bar{l}_a$ representation. On the other hand, we have shown that for a certain number of cases some representations of $\mathcal{H}$ contained in $\bar{l}_a$ cannot occur in the adjoint, like the momentum representation $\bar{l}_1$ for half the D and half the A cases. In this analysis we have made use of an auxiliary algebra $\mathcal{G}^{+++a}$. The conditions (2.10) and (2.12) are only necessary conditions and the actual outer mulitplicity μ of a given representation requires additional computations, which we have checked on a case by case basis. The reader may verify our statements by examining the tables in appendix A where we list the decompositions of $\mathcal{G}^{+++1}$ for a number of cases at low levels, for which we have also computed the outer multiplicities. In appendix A we also include the cases of E_6^{+++} and F_4^{+++} where we have applied a modified procedure to deal with the fact that there are the two nodes which we delete are joined by a line in the original Dynkin diagram.

2.2 Fundamentals in the adjoint representation?

In the above we considered the fundamental and adjoint representations of $\mathcal{G}^{+++}$ and answered the question of when the generators of one representation, when decomposed with respect to the subalgebra $\mathcal{H}$, occur in the other. To do this we considered the enlarged algebra $\mathcal{G}^{+++a}$ as a technical tool to find the generators in both representations. In particular, we allowed the relations between the two levels $n_c^{(1)}$ and $n_c^{(0)}$ to be arbitary.

We now ask a more restrictive question that the states in the lowest weight representation $\bar{l}_a$ of $\mathcal{G}^{+++}$ arise in the adjoint at the same level n_c as they occur in the lowest weight representation $\bar{l}_a$ of $\mathcal{G}^{+++}$. For simplicity, in this section we consider $\mathcal{G}^{+++}$ to have a symmetric Cartan matrix. We note from equation (2.6) that the lowest state in the $\bar{l}_a$ representation occurs at level $-\frac{(\nu,\lambda_a)}{x^2}$. The higher states occur at level $n_c^{(1)} - \frac{(\nu,\lambda_a)}{x^2}$. From equation (2.7) we find that the $\bar{l}_a$ representation can also be found at level $\mathcal{G}^{+++a}$ with $n_0 = 1$ and we see its states occur at level $n_c^{(1)} - \frac{(\nu,\lambda_a)}{x^2}$. We now demand that the corresponding representation arises in the adjoint representation at the same level

$$n_c^{(0)} = n_c^{(1)} - \frac{(\nu, \lambda_a)}{x^2}\,. \tag{2.43}$$

Substituting this into equation (2.15) we find it can be written as

$$n_k^{(0)} - n_k^{(1)} = -(A^{\mathcal{G}^{+++}})^{-1}_{ka}\,. \tag{2.44}$$

Hence we find the desired correspondence at all levels if $(A^{\mathcal{G}^{+++}})^{-1}_{ka}$ is a negative integer for all k. This is the case for $E_{10} = E_8^{++}$ as being self-dual its inverse Cartan matrix is integer valued and being hyperbolic it is also negative. One can verify that it is also satisfied for E_8^{+++} if a is even. It also works for $a = 9$.

Let us denote the roots of $\mathcal{G}^{+++a}$ corresponding to the $\bar{l}_a$ representation and the adjoint representation by $\beta^{(1)}$ and $\beta^{(0)}$ respectively. Using equation (2.12) we find that

$$(\beta^{(1)})^2 = 2 + (A^{\mathcal{G}^{+++}})^{-1}_{aa} + (\beta^{(0)})^2 \leq 2\,. \tag{2.45}$$

One must check this condition case by case. From (2.44) it is evident that $(A^{\mathcal{G}^{+++}})^{-1}_{ka}$ should be integral for all k. Analysing the case of E_8^{+++} where for even a the relevant row of the inverse Cartan matrix is integral we find the values $0, -4, -12, -24, -2$ for $a = 2, 4, 6, 8, 10$ respectively. This means that all states in the $\bar{l}_a$ representation of E_8^{+++} occur as solutions of the adjoint equations with length squared less than $2 - (A^{\mathcal{G}^{+++}})^{-1}_{aa}$. In particular, for $a = 2$ we have $(A^{\mathcal{G}^{+++}})^{-1}_{aa} = 0$ and so all fields deriving from imaginary roots $\beta^{(0)}$ of the adjoint of E_8^{+++} potentially are states of the l_2 representation. A similar argument with appropriate restrictions on the norm of $\beta^{(1)}$ we retrieve (some) fields of the $\bar{l}_a$ representation within the adjoint. This shows how to retrieve the remaining fundamental representations of E_8^{+++} within the adjoint of E_8^{+++}.

We stress that for general indefinite Kac-Moody algebras, unlike the finite dimensional, affine and hyperbolic case, it is not true that any (positive or negative) element β which has length squared less than or equal to 2 is a root of the algebra. The roots of a Kac-Moody algebra can be characterised as the union of the set of real roots and the set of imaginary roots. The real roots are those which are related to the simple roots by a Weyl transformation, and the imaginary roots are Weyl conjugate to elements in the fundamental Weyl chamber which have connected support on the Dynkin diagram [18]. This is to say that if one marks all points on the Dynkin diagram for which the components in the simple root expansion are non-zero, then this constitutes a connected subdiagram of the Dynkin diagram.

In appendix C we also give an example of the $\bar{l}_1$ representation of the hyperbolic $E_{10} = E_8^{++}$, where $E_8^{++1} = E_8^{+++}$ as already noted in [19]. We see that the identification of fields in the adjoint coming from imaginary roots with $\beta^2 \leq 0$ with fields in the $\bar{l}_1$ representation works well up to level $(12, 0)$ where for the first time we get a field in E_8^{++} which is not in the $\bar{l}_1$ representation. This is due to the subtlety of the roots of a general indefinite Kac-Moody algebra described above. As mentioned before, equations (2.10) and (2.12), or for that purpose (2.45) do not make any statement about the actual outer multiplicity μ of a given representation. It would be extremely informative to have a relation for these quantities as well, augmenting the relations (2.10) and (2.12). We stress again that the empirical data shows that the outer multiplicity is rarely vanishing, for instance within E_8^{++} we note that on the first twelve levels of E_8^{++} only $[1,0,0,0,0,0,0,0,0]$, $[0,0,0,0,0,0,0,1,0]$ and $[0,0,0,0,1,0,0,0,0]$ (interpreted as P_a, $Z^{a_1 a_2}$ and $Z^{a_1 \ldots a_5}$) and their multiples have vanishing outer multiplicity.

We remark that it cannot be the case that the adjoint representation contains the irreducible fundamental representation of $\mathcal{G}^{+++}$ within it as a sub-representation, but it might be the case that the fundamental representations of $\mathcal{G}^{+++}$, as they appear in the adjoint representation, are irreducible representations of a Borel subalgebra of $\mathcal{G}^{+++}$.

3. Interpretation of the $\bar{l}_1$ representation as "brane" charges

Much work has been carried out on the solutions of the ten and eleven dimensional supergravity theories which preserve a given fraction of supersymmetry (for a review see e.g. [20]). Those that preserve some of the 32 supersymmetries have topological charges

JHEP02(2004)033

that are related to the central charges that occur in the corresponding supersymmetry algebra [21]. For the p-branes that preserve half the supersymmetry the relation is particularly simple and they couple electrically or magnetically to the rank p topological central charges. The effective actions that describe the dynamics of the branes which preserve half the supersymmetry are required by supersymmetry to contain a coupling between the current corresponding to the central charge and background fields of the supergravity theory. Indeed, knowing the fields of a background supergravity theory one can read off the topological charges and so the 1/2 supersymmetric branes that occur in the given theory.

It was shown [9] that the l_1 representation of $E_{11} = E_8^{+++}$ contained the space-time translations and the two and five form central charges of the supersymmetry algebra together with an infinite number of higher level objects. This suggests that the extended theory, which possess an E_8^{+++} symmetry, should contain an infinite number of BPS solutions each of which has as its source a field in the theory and that the objects occuring in the l_1 representation are the corresponding charges. As such, one would expect that there should be a particular relationship between the fields of the theory and the objects in the l_1 representation. As every field in the theory is a Goldstone boson of E_8^{+++}, we expect that there is a correspondence between the generators in the adjoint representation of E_8^{+++} associated with the positive root space, and the objects in l_1. In this section we show that there is indeed such a correspondence and it is of the expected form.

In this section, for simpicity, we consider a $\mathcal{G}$ which is simply laced and for which $\mathcal{H}$ is of A type. We will show that given a $\mathcal{H}$ representation, with Dynkin indices $p_j^{(0)}$, that occurs in the adjoint representation of $\mathcal{G}^{+++}$ with $p_{l-1}^{(1)} \neq 0$, for some l, then there exist a representation of $\mathcal{H}$ in the $\bar{l}_1$ representation of $\mathcal{G}^{+++}$ with Dynkin indices p_j^1 given by

$$p_j^{(1)} - p_j^{(0)} = \begin{cases} 1\,, & j = l \\ -1\,, & j = l-1 \\ 0\,, & j \neq l, l-1\,. \end{cases} \tag{3.1}$$

For the simple case for which $p_l^{(1)} = 1$ we find $p_{l-1}^{(0)} = 1$, with all other Dynkin indices being zero. This means that if $\mathcal{G}^{+++}$ contains the object $R^{a_1 \cdots a_l}$ with corresponding field $A_{a_1 \ldots a_l}$, then the $\bar{l}_1$ representation of $\mathcal{G}^{+++}$ contains a generator $Z^{a_1 \cdots a_{l-1}}$.

The existence of the two representations of interest corresponds to the solutions of equations (2.10) and (2.12). Subtracting them we find that the condition for the $\mathcal{H}$ representation in $\bar{l}_1$ to appear in the adjoint is given by

$$\sum_j (\lambda_k, \lambda_j)(p_j^{(1)} - p_j^{(0)}) = (\lambda_1, \lambda_k) - (n_k^{(1)} - n_k^{(0)}) \tag{3.2}$$

Using the correspondence of equation (3.1) we find that the above equation becomes

$$(\lambda_l - \lambda_{l-1} - \lambda_1, \lambda_k) = n_k^{(0)} - n_k^{(1)}\,. \tag{3.3}$$

Recognising the $\lambda_{l-1} + \lambda_1 - \lambda_l$ is just $\alpha_1 + \ldots + \alpha_{l-1}$, we find that it is always true provided

$$n_k^{(1)} - n_k^{(0)} = \begin{cases} 1\,, & k \leq l-1\,, \\ 0\,, & k \geq l\,. \end{cases} \tag{3.4}$$

JHEP02(2004)033

We note that if $n_k^{(0)}$ is positive then $n_k^{(1)}$ is automatically positive and so the representation always occurs. Here we can also say something about the outer multiplicity: One can see that any $\mathcal{H}$-representation within $\mathcal{G}^{+++}$ lifts to a corresponding one in $\mathcal{G}^{+++1}$ under the $\mathcal{H}$ extended by one node to the left. This representation then will reduce to the original one of $\mathcal{G}^{+++}$ and one with one box filled in with a 0, that is with the rank decreased by one. Therefore it is a corollary of the result that the very-extended $\mathcal{G}^{+++}$ theories contain at least the fields of the oxidised theories [16], that, if augmented by the $\bar{l}_1$ representation, they also contain all the correct central charges for these fields.

We now show independently that the additional condition of equation (2.12) is satisfied for the $\bar{l}_1$ representation, $n^{(0)} = 1$, if it is satisfied for the adjoint representation, $n^{(0)} = 1$. Let us denote the length squared of the roots in $\mathcal{G}^{+++1}$ corresponding to the $\bar{l}_1$ representation and the adjoint representation by $\beta^{(1)}$ and $\beta^{(0)}$ respectively. Using equation (2.12) we find that

$$(\beta^{(1)})^2 - (\beta^{(0)})^2 = y^2 + \frac{(\nu,\lambda_1)^2}{x^2} - 2n_c(\nu,\lambda_1) + \sum_{ij} p_i^{(1)} A_{ij}^{-1} p_j^{(1)} - \sum_{ij} p_i^{(0)} A_{ij}^{-1} p_j^{(0)} \,. \quad (3.5)$$

Using equation (2.9) and the relation $y^2 + l_1^2 = 2$ we find that

$$(\beta^{(1)})^2 - (\beta^{(0)})^2 = 2 - n_k^{(1)} - n_k^{(0)} - \sum_j (n_j^{(1)} - n_j^{(0)})(p_j^{(1)} + p_j^{(0)}) \,. \quad (3.6)$$

Finally, using equations (3.1) and (3.4) we conclude that the left-hand side is a negative integer and so if $(\beta^{(0)})^2 = 2, 0, \ldots$ then $\beta^{(1)}$ has an acceptable length squared.

4. Where is space-time?

As we have explained in the introduction there is growing evidence for the conjecture [4] that an extension of eleven dimensional supergravity possesses an E_8^{+++} symmetry which is non-linearly realised. Reference [4] did not address the way space-time should enter the construction, but there are now essentially two different proposals as to how this should arise.

It has been proposed [6] that the fields of the non-linear realisation of E_{10} should depend only on time and the spatial dependence should occur through the equations of motion of the non-linear realisation, i.e. the spatial dependence should essentially arise as a result of the properties of the adjoint representation of E_8^{++}. Alternatively, it has been proposed [9] that one should form the non-linear realisation of the semi-direct product of E_8^{+++} with the $\bar{l}_1$ representation in both cases. In this case the fields would depend on the coordinates introduced into the non-linear realisation by the $\bar{l}_1$ representation in both cases. These would include space-time, coordinates for the central charges of the eleven dimensional supersymmetry algebra and an infinite number of other coordinates. There is also the suggestion of reference [10] in which a non-linear realisation of E_8^{+++} is considered, but the fields depend on an auxiliary parameter and it is hoped that the space-time dependence occurs by a mechanism similar to that proposed in reference [6]. These authors also suggested that the level seven generator in E_8^{+++}, which has the correct A_{10} properties, i.e. $p_1 = 1$, be identified with the space-time translations.

JHEP02(2004)033

In [6] it was shown that for the 3-form, 6-form and dual graviton representations at levels 1, 2 and 3 of E_8^{++} there are tensors with the index structures of a k-th spatial derivative at levels $1+3k$, $2+3k$ and $3+3k$ respectively. It is hoped [6] that the dynamics will imply that the fields corresponding to the latter generators will turn out to be the spatial derivatives of fields correponding to the former generators. In fact, the roots giving rise to these k-th derivative fields are consistent with the action of a generator with $p_1 = 1$ at level 3 which satifies the constraints of equations (2.10) and (2.12). This element at level three is just the fundamental lowest weight of the over extended node and so identical with the affine root of the E_8^+ subalgebra. As discussed in section (2.2), this is just the first object in the infinite tower of the $\bar{l}_1$ representation. One might interpret this level 3 generator as a space-time generator, but as it has outer multiplicity zero it does not appear in the E_8^{++} algebra.

It was conjectured [10] that the adjoint representation for all $\mathcal{G}^{+++}$, contains a generator that belongs to the correct A representation, i.e. $p_1 = 1,\ p_j = 0,\ j \geq 2$, to be identified as space-time translations. However, we have seen that this is not the case for several important examples, namely D_{n-3}^{+++} for odd n and A_{n-3}^{+++} for even n.[1] As was explained in [5, 16], there exist two alternative choices for a regular A_9 subalgebra of E_8^{+++} and the decompositions there give rise to the fields of the two type-II theories at lowest levels. It is not hard to see that taking the IIB decomposition of E_8^{+++} there can be no object with the structure of a space-time translation operator in the spectrum. For these reason, the mechanism suggested in [10] can not apply to all $\mathcal{G}^{+++}$ and also fails in particular for the interesting cases of type-IIB theory and the bosonic string D_{24}^{+++}.

Furthermore, even when a generator with the correct A representation property does occur it is usually not unique. For example, as we found in section two, for the case of E_8^{+++}, such a generator occurs at levels $7+11s$ for any non-negative integer s and so one can wonder which of these is the real space-time translation generator. A similar result occurs for E_8^{++}, where potential space-time operators occur at level $3+10r$ for any non-negative integer r. To make the situation more confusing one finds that some of these operators occur with outer multiplicity greater than one and so even at a given level there is often more than one choice. For E_8^{++} the corresponding statement is that there are P_a-like operators at levels $3+10r$ with r a non-negative integer, and again for $r \geq 1$ it seems that they have non-vanishing outer multiplicity [11].

Using the analysis of section two, one can show that given a representation of A_{10} with Dynkin indices p_j, at level n_c, in the adjoint representation of E_8^{+++}, then the adjoint representation also contains a representation with Dynkin indicies $p_1+m, p_j, j \geq 2$ at level $n_c + m(7+11s)$. Clearly, the field associated to the latter generator in the non-linear realisation has the correct A_{10} character to be identified with the space-time derivatives of the field associated with the former generator. A similar result holds for E_8^{++} and the possible space momentum generators mentioned above. Hence from this perspective the possible momentum generators are on an equal footing.

[1] This result was communicated to the authors of reference [10] who have subsequently changed their paper to take account of these results.

Even if a generator has the correct A properties to be identified with a space-time generator P_a one can wonder if its commutator vanishes. For the level seven candidate of E_8^{+++}, suggested in reference [10], this is unlikely to be the case as at level 14 there exist a $p_2 = 1$ representation that has a root of precisely the correct type to occur on the right-hand side of the $[P_a, P_b]$ commutator. This would be inconsistent with our usual understanding of what is meant by a space-time translation operator, and it would imply that the theory has some kind of non-commutative structure. We note that this does not apply to the level 3 operator of E_8^{++} mentioned above as its outer multiplicitly is zero and so does not actually occur in the theory.

As we show in section two when a generator with Dynkin indices $p_1 = 1,\ p_j = 0,\ j \geq 2$ does occur in the adjoint representation of $\mathcal{G}^{+++}$, it can usually be thought of as part of an infinite tower associated with the $\bar{l}_1$ representation. For the case of E_8^{+++} the next object, Z^{ab}, in the tower has $p_9 = 2,\ p_j = 0,\ j \neq 9$ and occurs at levels $n_c = 8 + 11k$ ($k \geq 0$, for $k = 0$ it has outer multiplicity 5). Given a representation Dynkin indices p_j at level n_c in the adjoint representation of E_8^{+++}, then the adjoint representation also contains a solution to (2.10) and (2.12) with Dynkin indices $p_9 + m, p_j, j \neq 9$ at level $n_c + s(11k + 8)$, for non-negative integer s. The fields associated with the latter generator have the correct structure to be interpreted as derivatives with repect to Z^{ab}. One might hope that this is also a consequence of the dynamics. Hence, one has in effect a theory where the fields depend on space-time and additional coordinates associated with the central charges. The same is likely to hold for all the objects of the $\bar{l}_1$ representation and as such even if the derivatives arise from within E_8^{+++} the $\bar{l}_1$ representation is likely to play an important role. A similar result also holds for E_8^{++}.

The introduction of space-time by considering the semi-direct product of E_8^{++} with the $\bar{l}_1$ representation may appear less elegant, however, as we have explained the $\bar{l}_1$ representation arises naturally in the context of $E_{12} = E_8^{+++1}$ and the roots with level $n_0 = 1$. There may also be reasons associated with the conformal symmetry found in reference [3, 9] to also introduce the level $n_0 = -1$ roots which is the dual highest weight l_1 representation. The levels $n_0 = -1, 0, 1$ form what is known as the local subalgebra of $\mathcal{G}^{+++a}$ and the lowest n_c level fields just form an algebra resembling the conformal group with the only difference being that the Lorentz generators J_{ab} are traded in for the generators $K^a{}_b$ of the general linear group (which has its Cartan invariant subalgebra has the Lorentz group).[2] In the supersymmetric extension one knows [3] that the conformal group gets enlarged to an algebra which must contain $Osp(1|64)$ [3] which in turn has SL(32) as a subalgebra. This SL(32) is contained in the Cartan invariant subalgebra of E_8^{+++} as a truncation, but not as a subalgebra [9] and hence it is understandable that the conformal algebra gets enlarged in the way suggested by adding the l_1 representation. We also note that if we apply the diagram extension to the semi-direct product of the Lorentz group with a momentum

[2] In [4] the local subalgebra taken for E_8^{+++} was that which was invariant under the Cartan involution $E_a \leftrightarrow -F_a, H_a \rightarrow -H_a$. Considering the gravity line SL(n) one finds that the corresponding local subalgebra is SO(n) and not SO($1, n-1$). As a result, to arrive at Minkowski space one must carry out a Wick rotation. To arrive directly in Minkowski space one can take the above involution except for $E_1 \leftrightarrow F_1$. A modification along these lines was presented in [10], but the involution taken appears not to be the same.

generator, i.e. the Poincaré group, then one finds the conformal algebra. In fact, the momentum generators and all the central charges commute in this local part of $\mathcal{G}^{+++1}$. If one considers the whole of $\mathcal{G}^{+++1}$, then the momentum generators will still commute but this is then no longer necessarily true for the higher charges. However, if one also includes the $\bar{l}_2$ representation, even the P_a operators no longer need to commute. Hence, it is possible that the semi-direct product is part of some larger algebra.

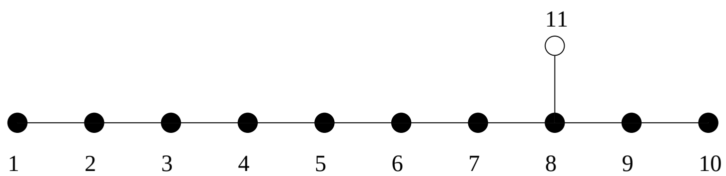
It is clear that how space-time enters is intricately connected with the representation theory of E_{11} and in this paper we have tried to illuminate this connection. Clearly, further work is required on this fascinating problem.

Acknowledgments

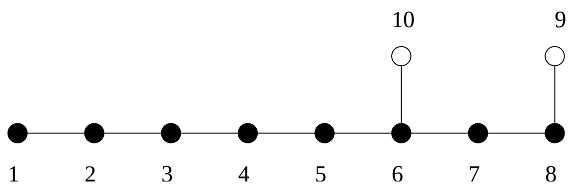
AK would like to thank the Studienstiftung des deutschen Volkes for financial support and King's College, London, for hospitality. This research was supported in part by the PPARC grants PPA/G/O/2000/00451 and PPA/G/S4/1998/00613.

A. Dynkin diagrams for the very extended algebras

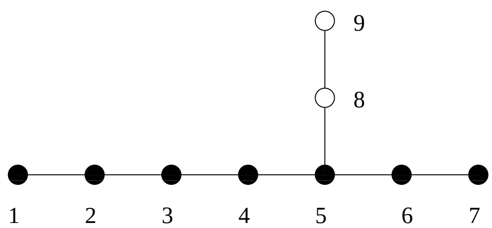
- E_8^{+++}



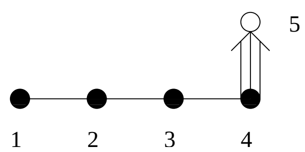
- E_7^{+++}



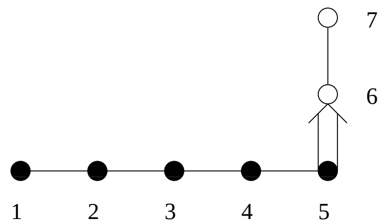
- E_6^{+++}



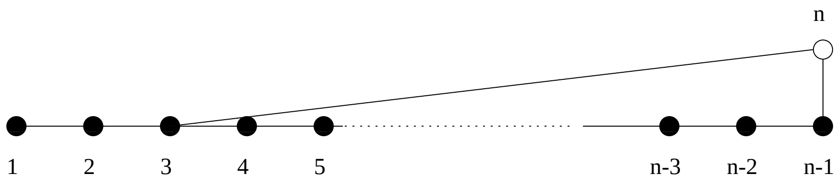
- G_2^{+++}



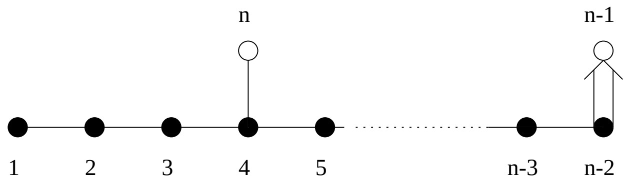
- F_4^{+++}



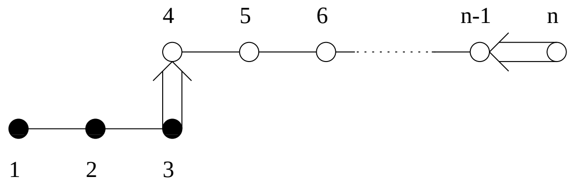
- A_{n-3}^{+++}



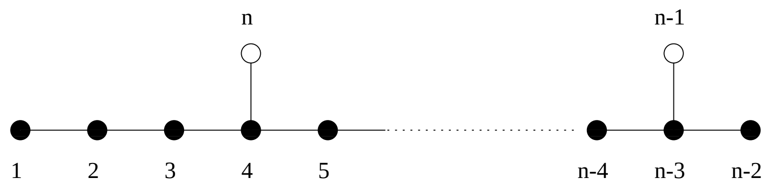
- B_{n-3}^{+++}



- C_{n-3}^{+++}



- D_{n-3}^{+++}



B. Tables for the $\bar{l}_1$ representation of various $\mathcal{G}^{+++}$

B.1 E_8^{+++1}

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,0)	[0,0,0,0,0,0,0,1,0,0]	(0,0,0,0,0,0,0,0,0,0,0,1)	2	1	1
(2,0)	[0,0,0,0,1,0,0,0,0,0]	(0,0,0,0,0,0,1,2,3,2,1,2)	2	11	1
(3,0)	[0,0,1,0,0,0,0,0,0,1]	(0,0,0,0,1,2,3,4,5,3,1,3)	2	22	1
(3,0)	[0,1,0,0,0,0,0,0,0,0]	(0,0,0,1,2,3,4,5,6,4,2,3)	0	30	0
(4,0)	[0,1,0,0,0,0,0,1,0,0]	(0,0,0,1,2,3,4,5,6,4,2,4)	2	31	1

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(4,0)	[1,0,0,0,0,0,0,0,0,2]	(0,0,1,2,3,4,5,6,7,4,1,4)	2	37	1
(4,0)	[1,0,0,0,0,0,0,0,1,0]	(0,0,1,2,3,4,5,6,7,4,2,4)	0	38	0
(4,0)	[0,0,0,0,0,0,0,0,0,1]	(0,1,2,3,4,5,6,7,8,5,2,4)	-2	47	1
(5,0)	[1,0,0,0,0,0,1,0,0,1]	(0,0,1,2,3,4,5,6,8,5,2,5)	2	41	1
(5,0)	[0,1,0,0,1,0,0,0,0,0]	(0,0,0,1,2,3,5,7,9,6,3,5)	2	41	1
(5,0)	[1,0,0,0,0,1,0,0,0,0]	(0,0,1,2,3,4,5,7,9,6,3,5)	0	45	0
(5,0)	[0,0,0,0,0,0,0,1,0,1]	(0,1,2,3,4,5,6,7,8,5,2,5)	0	48	1
(5,0)	[0,0,0,0,0,0,1,0,0,0]	(0,1,2,3,4,5,6,7,9,6,3,5)	-2	51	1
(6,0)	[1,0,0,0,1,0,0,0,1,0]	(0,0,1,2,3,4,6,8,10,6,3,6)	2	49	1
(6,0)	[0,1,1,0,0,0,0,0,0,1]	(0,0,0,1,3,5,7,9,11,7,3,6)	2	52	1
(6,0)	[0,0,0,0,0,0,1,1,0,0]	(0,1,2,3,4,5,6,7,9,6,3,6)	2	52	1
(6,0)	[0,0,0,0,0,1,0,0,0,2]	(0,1,2,3,4,5,6,8,10,6,2,6)	2	53	1
(6,0)	[1,0,0,1,0,0,0,0,0,1]	(0,0,1,2,3,5,7,9,11,7,3,6)	0	54	1
(6,0)	[0,0,0,0,0,1,0,0,1,0]	(0,1,2,3,4,5,6,8,10,6,3,6)	0	54	0
(6,0)	[0,0,0,0,1,0,0,0,0,1]	(0,1,2,3,4,5,7,9,11,7,3,6)	-2	58	2
(6,0)	[0,2,0,0,0,0,0,0,0,0]	(0,0,0,2,4,6,8,10,12,8,4,6)	0	60	0
(6,0)	[1,0,1,0,0,0,0,0,0,0]	(0,0,1,2,4,6,8,10,12,8,4,6)	-2	61	1
(6,0)	[0,0,0,1,0,0,0,0,0,0]	(0,1,2,3,4,6,8,10,12,8,4,6)	-4	64	1
(7,0)	[1,0,0,1,0,0,1,0,0,0]	(0,0,1,2,3,5,7,9,12,8,4,7)	2	58	1
(7,0)	[0,0,0,0,1,0,0,1,0,1]	(0,1,2,3,4,5,7,9,11,7,3,7)	2	59	1
(7,0)	[1,0,1,0,0,0,0,0,1,1]	(0,0,1,2,4,6,8,10,12,7,3,7)	2	60	1
(7,0)	[0,2,0,0,0,0,0,1,0,0]	(0,0,0,2,4,6,8,10,12,8,4,7)	2	61	1
(7,0)	[1,0,1,0,0,0,0,1,0,0]	(0,0,1,2,4,6,8,10,12,8,4,7)	0	62	1
(7,0)	[0,0,0,0,1,0,1,0,0,0]	(0,1,2,3,4,5,7,9,12,8,4,7)	0	62	1
(7,0)	[0,0,0,1,0,0,0,0,1,1]	(0,1,2,3,4,6,8,10,12,7,3,7)	0	63	1
(7,0)	[0,0,0,1,0,0,0,1,0,0]	(0,1,2,3,4,6,8,10,12,8,4,7)	-2	65	2
(7,0)	[1,1,0,0,0,0,0,0,0,2]	(0,0,1,3,5,7,9,11,13,8,3,7)	0	67	1
(7,0)	[1,1,0,0,0,0,0,0,1,0]	(0,0,1,3,5,7,9,11,13,8,4,7)	-2	68	2
(7,0)	[0,0,1,0,0,0,0,0,0,2]	(0,1,2,3,5,7,9,11,13,8,3,7)	-2	69	3
(7,0)	[0,0,1,0,0,0,0,0,1,0]	(0,1,2,3,5,7,9,11,13,8,4,7)	-4	70	3
(7,0)	[2,0,0,0,0,0,0,0,0,1]	(0,0,2,4,6,8,10,12,14,9,4,7)	-4	76	1
(7,0)	[0,1,0,0,0,0,0,0,0,1]	(0,1,2,4,6,8,10,12,14,9,4,7)	-6	77	4
(7,0)	[1,0,0,0,0,0,0,0,0,0]	(0,1,3,5,7,9,11,13,15,10,5,7)	-8	86	1
(8,0)	[1,0,1,0,0,1,0,0,0,1]	(0,0,1,2,4,6,8,11,14,9,4,8)	2	67	1
(8,0)	[0,0,0,1,0,0,1,0,1,0]	(0,1,2,3,4,6,8,10,13,8,4,8)	2	67	1
(8,0)	[1,1,0,0,0,0,0,1,1,0]	(0,0,1,3,5,7,9,11,13,8,4,8)	2	69	1
(8,0)	[0,0,0,0,2,0,0,0,0,1]	(0,1,2,3,4,5,8,11,14,9,4,8)	2	69	1
(8,0)	[0,0,1,0,0,0,0,1,0,2]	(0,1,2,3,5,7,9,11,13,8,3,8)	2	70	1
(8,0)	[0,0,0,1,0,1,0,0,0,1]	(0,1,2,3,4,6,8,11,14,9,4,8)	0	70	1
(8,0)	[1,1,0,0,0,0,1,0,0,1]	(0,0,1,3,5,7,9,11,14,9,4,8)	0	71	2
(8,0)	[1,0,0,2,0,0,0,0,0,0]	(0,0,1,2,3,6,9,12,15,10,5,8)	2	71	1

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(8,0)	[0,2,0,0,1,0,0,0,0,0]	(0,0,0,2,4,6,9,12,15,10,5,8)	2	71	1
(8,0)	[0,0,1,0,0,0,0,1,1,0]	(0,1,2,3,5,7,9,11,13,8,4,8)	0	71	1
(8,0)	[1,0,1,0,1,0,0,0,0,0]	(0,0,1,2,4,6,9,12,15,10,5,8)	0	72	1
(8,0)	[0,0,1,0,0,0,1,0,0,1]	(0,1,2,3,5,7,9,11,14,9,4,8)	-2	73	4
(8,0)	[2,0,0,0,0,0,0,0,1,2]	(0,0,2,4,6,8,10,12,14,8,3,8)	2	75	1
(8,0)	[1,1,0,0,0,1,0,0,0,0]	(0,0,1,3,5,7,9,12,15,10,5,8)	-2	75	2
(8,0)	[0,0,0,1,1,0,0,0,0,0]	(0,1,2,3,4,6,9,12,15,10,5,8)	-2	75	2
(8,0)	[2,0,0,0,0,0,0,0,2,0]	(0,0,2,4,6,8,10,12,14,8,4,8)	0	76	0
(8,0)	[0,1,0,0,0,0,0,0,1,2]	(0,1,2,4,6,8,10,12,14,8,3,8)	0	76	1
(8,0)	[2,0,0,0,0,0,0,1,0,1]	(0,0,2,4,6,8,10,12,14,9,4,8)	-2	77	2
(8,0)	[0,1,0,0,0,0,0,0,2,0]	(0,1,2,4,6,8,10,12,14,8,4,8)	-2	77	2
(8,0)	[0,0,1,0,0,1,0,0,0,0]	(0,1,2,3,5,7,9,12,15,10,5,8)	-4	77	3
(8,0)	[0,1,0,0,0,0,0,1,0,1]	(0,1,2,4,6,8,10,12,14,9,4,8)	-4	78	6
(8,0)	[2,0,0,0,0,0,1,0,0,0]	(0,0,2,4,6,8,10,12,15,10,5,8)	-4	80	2
(8,0)	[0,1,0,0,0,0,1,0,0,0]	(0,1,2,4,6,8,10,12,15,10,5,8)	-6	81	7
(8,0)	[1,0,0,0,0,0,0,0,0,3]	(0,1,3,5,7,9,11,13,15,9,3,8)	-2	84	2
(8,0)	[1,0,0,0,0,0,0,0,1,1]	(0,1,3,5,7,9,11,13,15,9,4,8)	-6	85	7
(8,0)	[1,0,0,0,0,0,0,1,0,0]	(0,1,3,5,7,9,11,13,15,10,5,8)	-8	87	6
(8,0)	[0,0,0,0,0,0,0,0,0,2]	(0,2,4,6,8,10,12,14,16,10,4,8)	-8	94	3
(8,0)	[0,0,0,0,0,0,0,0,1,0]	(0,2,4,6,8,10,12,14,16,10,5,8)	-10	95	5
(0,1)	[1,0,0,0,0,0,0,0,0,0]	(1,0,0,0,0,0,0,0,0,0,0,0)	2	1	1
(1,1)	[0,0,0,0,0,0,0,0,1,0]	(1,1,1,1,1,1,1,1,1,0,0,1)	2	10	1
(2,1)	[0,0,0,0,0,1,0,0,0,0]	(1,1,1,1,1,1,1,2,3,2,1,2)	2	17	1
(3,1)	[0,0,0,1,0,0,0,0,0,1]	(1,1,1,1,1,2,3,4,5,3,1,3)	2	26	1
(3,1)	[0,0,1,0,0,0,0,0,0,0]	(1,1,1,1,2,3,4,5,6,4,2,3)	0	33	1
(4,1)	[0,0,1,0,0,0,0,1,0,0]	(1,1,1,1,2,3,4,5,6,4,2,4)	2	34	1
(4,1)	[0,1,0,0,0,0,0,0,0,2]	(1,1,1,2,3,4,5,6,7,4,1,4)	2	39	1
(4,1)	[0,1,0,0,0,0,0,0,1,0]	(1,1,1,2,3,4,5,6,7,4,2,4)	0	40	1
(4,1)	[1,0,0,0,0,0,0,0,0,1]	(1,1,2,3,4,5,6,7,8,5,2,4)	-2	48	2
(4,1)	[0,0,0,0,0,0,0,0,0,0]	(1,2,3,4,5,6,7,8,9,6,3,4)	-4	58	1
(5,1)	[0,1,0,0,0,0,1,0,0,1]	(1,1,1,2,3,4,5,6,8,5,2,5)	2	43	1
(5,1)	[0,0,1,0,1,0,0,0,0,0]	(1,1,1,1,2,3,5,7,9,6,3,5)	2	44	1
(5,1)	[0,1,0,0,0,1,0,0,0,0]	(1,1,1,2,3,4,5,7,9,6,3,5)	0	47	1
(5,1)	[1,0,0,0,0,0,0,1,0,1]	(1,1,2,3,4,5,6,7,8,5,2,5)	0	49	2
(5,1)	[1,0,0,0,0,0,1,0,0,0]	(1,1,2,3,4,5,6,7,9,6,3,5)	-2	52	2
(5,1)	[0,0,0,0,0,0,0,0,0,3]	(1,2,3,4,5,6,7,8,9,5,1,5)	2	56	1
(5,1)	[0,0,0,0,0,0,0,0,1,1]	(1,2,3,4,5,6,7,8,9,5,2,5)	-2	57	2
(5,1)	[0,0,0,0,0,0,0,1,0,0]	(1,2,3,4,5,6,7,8,9,6,3,5)	-4	59	3
(6,1)	[0,1,0,0,1,0,0,0,1,0]	(1,1,1,2,3,4,6,8,10,6,3,6)	2	51	1
(6,1)	[1,0,0,0,0,0,1,1,0,0]	(1,1,2,3,4,5,6,7,9,6,3,6)	2	53	1
(6,1)	[1,0,0,0,0,1,0,0,0,2]	(1,1,2,3,4,5,6,8,10,6,2,6)	2	54	1

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(6,1)	[1,0,0,0,0,1,0,0,1,0]	(1,1,2,3,4,5,6,8,10,6,3,6)	0	55	1
(6,1)	[0,0,2,0,0,0,0,0,0,1]	(1,1,1,1,3,5,7,9,11,7,3,6)	2	55	1
(6,1)	[0,1,0,1,0,0,0,0,0,1]	(1,1,1,2,3,5,7,9,11,7,3,6)	0	56	2
(6,1)	[1,0,0,0,1,0,0,0,0,1]	(1,1,2,3,4,5,7,9,11,7,3,6)	-2	59	4
(6,1)	[0,0,0,0,0,0,1,0,0,2]	(1,2,3,4,5,6,7,8,10,6,2,6)	0	60	2
(6,1)	[0,0,0,0,0,0,0,2,0,0]	(1,2,3,4,5,6,7,8,9,6,3,6)	0	60	1
(6,1)	[0,0,0,0,0,0,1,0,1,0]	(1,2,3,4,5,6,7,8,10,6,3,6)	-2	61	3
(6,1)	[0,1,1,0,0,0,0,0,0,0]	(1,1,1,2,4,6,8,10,12,8,4,6)	-2	63	2
(6,1)	[0,0,0,0,0,1,0,0,0,1]	(1,2,3,4,5,6,7,9,11,7,3,6)	-4	64	5
(6,1)	[1,0,0,1,0,0,0,0,0,0]	(1,1,2,3,4,6,8,10,12,8,4,6)	-4	65	3
(6,1)	[0,0,0,0,1,0,0,0,0,0]	(1,2,3,4,5,6,8,10,12,8,4,6)	-6	69	5
(7,1)	[1,0,0,0,1,0,0,1,0,1]	(1,1,2,3,4,5,7,9,11,7,3,7)	2	60	1
(7,1)	[0,1,0,1,0,0,1,0,0,0]	(1,1,1,2,3,5,7,9,12,8,4,7)	2	60	1
(7,1)	[0,1,1,0,0,0,0,0,1,1]	(1,1,1,2,4,6,8,10,12,7,3,7)	2	62	1
(7,1)	[1,0,0,0,1,0,1,0,0,0]	(1,1,2,3,4,5,7,9,12,8,4,7)	0	63	2
(7,1)	[1,0,0,1,0,0,0,0,1,1]	(1,1,2,3,4,6,8,10,12,7,3,7)	0	64	2
(7,1)	[0,1,1,0,0,0,0,1,0,0]	(1,1,1,2,4,6,8,10,12,8,4,7)	0	64	2
(7,1)	[0,0,0,0,0,0,2,0,0,1]	(1,2,3,4,5,6,7,8,11,7,3,7)	2	64	1
(7,1)	[0,0,0,0,0,1,0,1,0,1]	(1,2,3,4,5,6,7,9,11,7,3,7)	0	65	2
(7,1)	[1,0,0,1,0,0,0,1,0,0]	(1,1,2,3,4,6,8,10,12,8,4,7)	-2	66	4
(7,1)	[0,0,0,0,1,0,0,0,0,3]	(1,2,3,4,5,6,8,10,12,7,2,7)	2	67	1
(7,1)	[0,0,0,0,1,0,0,0,1,1]	(1,2,3,4,5,6,8,10,12,7,3,7)	-2	68	4
(7,1)	[0,0,0,0,0,1,1,0,0,0]	(1,2,3,4,5,6,7,9,12,8,4,7)	-2	68	2
(7,1)	[0,2,0,0,0,0,0,0,0,2]	(1,1,1,3,5,7,9,11,13,8,3,7)	0	69	1
(7,1)	[1,0,1,0,0,0,0,0,0,2]	(1,1,2,3,5,7,9,11,13,8,3,7)	-2	70	5
(7,1)	[0,2,0,0,0,0,0,0,1,0]	(1,1,1,3,5,7,9,11,13,8,4,7)	-2	70	3
(7,1)	[0,0,0,0,1,0,0,1,0,0]	(1,2,3,4,5,6,8,10,12,8,4,7)	-4	70	7
(7,1)	[1,0,1,0,0,0,0,0,1,0]	(1,1,2,3,5,7,9,11,13,8,4,7)	-4	71	7
(7,1)	[0,0,0,1,0,0,0,0,0,2]	(1,2,3,4,5,7,9,11,13,8,3,7)	-4	73	7
(7,1)	[0,0,0,1,0,0,0,0,1,0]	(1,2,3,4,5,7,9,11,13,8,4,7)	-6	74	11
(7,1)	[1,1,0,0,0,0,0,0,0,1]	(1,1,2,4,6,8,10,12,14,9,4,7)	-6	78	8
(7,1)	[0,0,1,0,0,0,0,0,0,1]	(1,2,3,4,6,8,10,12,14,9,4,7)	-8	80	15
(7,1)	[2,0,0,0,0,0,0,0,0,0]	(1,1,3,5,7,9,11,13,15,10,5,7)	-8	87	2
(7,1)	[0,1,0,0,0,0,0,0,0,0]	(1,2,3,5,7,9,11,13,15,10,5,7)	-10	88	8

B.2 E_7^{+++1}

(n_c, n_d, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,0,0)	[0,0,0,0,0,1,0,0]	(0,0,0,0,0,0,0,0,0,0,1)	2	1	1
(2,0,0)	[0,0,1,0,0,0,0,0]	(0,0,0,0,1,2,3,2,1,0,2)	2	11	1
(3,0,0)	[1,0,0,0,0,0,0,1]	(0,0,1,2,3,4,5,3,1,0,3)	2	22	1
(3,0,0)	[0,0,0,0,0,0,0,0]	(0,1,2,3,4,5,6,4,2,0,3)	0	30	0

JHEP02(2004)033

(n_c,n_d,n_0)	p_i	β	β^2	$ht(\beta)$	μ
(4,0,0)	[0,0,0,0,0,1,0,0]	(0,1,2,3,4,5,6,4,2,0,4)	2	31	1
(5,0,0)	[0,0,1,0,0,0,0,0]	(0,1,2,3,5,7,9,6,3,0,5)	2	41	1
(0,1,0)	[0,0,0,0,0,0,0,1]	(0,0,0,0,0,0,0,0,0,1,0)	2	1	1
(1,1,0)	[0,0,0,0,1,0,0,0]	(0,0,0,0,0,0,1,1,1,1,1)	2	5	1
(2,1,0)	[0,0,1,0,0,0,0,1]	(0,0,0,0,1,2,3,2,1,1,2)	2	12	1
(2,1,0)	[0,1,0,0,0,0,0,0]	(0,0,0,1,2,3,4,3,2,1,2)	0	18	1
(3,1,0)	[0,1,0,0,0,1,0,0]	(0,0,0,1,2,3,4,3,2,1,3)	2	19	1
(3,1,0)	[1,0,0,0,0,0,0,2]	(0,0,1,2,3,4,5,3,1,1,3)	2	23	1
(3,1,0)	[1,0,0,0,0,0,1,0]	(0,0,1,2,3,4,5,3,2,1,3)	0	24	1
(3,1,0)	[0,0,0,0,0,0,0,1]	(0,1,2,3,4,5,6,4,2,1,3)	-2	31	2
(4,1,0)	[1,0,0,0,1,0,0,1]	(0,0,1,2,3,4,6,4,2,1,4)	2	27	1
(4,1,0)	[0,1,1,0,0,0,0,0]	(0,0,0,1,3,5,7,5,3,1,4)	2	29	1
(4,1,0)	[1,0,0,1,0,0,0,0]	(0,0,1,2,3,5,7,5,3,1,4)	0	31	1
(4,1,0)	[0,0,0,0,0,1,0,1]	(0,1,2,3,4,5,6,4,2,1,4)	0	32	2
(4,1,0)	[0,0,0,0,1,0,0,0]	(0,1,2,3,4,5,7,5,3,1,4)	-2	35	2
(5,1,0)	[1,0,1,0,0,0,1,0]	(0,0,1,2,4,6,8,5,3,1,5)	2	35	1
(5,1,0)	[0,0,0,0,1,1,0,0]	(0,1,2,3,4,5,7,5,3,1,5)	2	36	1
(5,1,0)	[0,0,0,1,0,0,0,2]	(0,1,2,3,4,6,8,5,2,1,5)	2	37	1
(5,1,0)	[0,0,0,1,0,0,1,0]	(0,1,2,3,4,6,8,5,3,1,5)	0	38	1
(5,1,0)	[1,1,0,0,0,0,0,1]	(0,0,1,3,5,7,9,6,3,1,5)	0	40	2
(5,1,0)	[0,0,1,0,0,0,0,1]	(0,1,2,3,5,7,9,6,3,1,5)	-2	42	4
(5,1,0)	[2,0,0,0,0,0,0,0]	(0,0,2,4,6,8,10,7,4,1,5)	-2	47	1
(5,1,0)	[0,1,0,0,0,0,0,0]	(0,1,2,4,6,8,10,7,4,1,5)	-4	48	3
(2,2,0)	[0,1,0,0,0,0,0,1]	(0,0,0,1,2,3,4,3,2,2,2)	2	19	1
(2,2,0)	[1,0,0,0,0,0,0,0]	(0,0,1,2,3,4,5,4,3,2,2)	0	26	0
(3,2,0)	[0,1,0,0,1,0,0,0]	(0,0,0,1,2,3,5,4,3,2,3)	2	23	1
(3,2,0)	[1,0,0,0,0,0,1,1]	(0,0,1,2,3,4,5,3,2,2,3)	2	25	1
(3,2,0)	[1,0,0,0,0,1,0,0]	(0,0,1,2,3,4,5,4,3,2,3)	0	27	1
(3,2,0)	[0,0,0,0,0,0,0,2]	(0,1,2,3,4,5,6,4,2,2,3)	0	32	1
(3,2,0)	[0,0,0,0,0,0,1,0]	(0,1,2,3,4,5,6,4,3,2,3)	-2	33	2
(4,2,0)	[1,0,0,0,1,0,1,0]	(0,0,1,2,3,4,6,4,3,2,4)	2	29	1
(4,2,0)	[0,1,1,0,0,0,0,1]	(0,0,0,1,3,5,7,5,3,2,4)	2	30	1
(4,2,0)	[1,0,0,1,0,0,0,1]	(0,0,1,2,3,5,7,5,3,2,4)	0	32	2
(4,2,0)	[0,0,0,0,0,1,0,2]	(0,1,2,3,4,5,6,4,2,2,4)	2	33	1
(4,2,0)	[0,0,0,0,0,1,1,0]	(0,1,2,3,4,5,6,4,3,2,4)	0	34	1
(4,2,0)	[0,2,0,0,0,0,0,0]	(0,0,0,2,4,6,8,6,4,2,4)	0	36	1
(4,2,0)	[0,0,0,0,1,0,0,1]	(0,1,2,3,4,5,7,5,3,2,4)	-2	36	4
(4,2,0)	[1,0,1,0,0,0,0,0]	(0,0,1,2,4,6,8,6,4,2,4)	-2	37	3
(4,2,0)	[0,0,0,1,0,0,0,0]	(0,1,2,3,4,6,8,6,4,2,4)	-4	40	3
(0,0,1)	[1,0,0,0,0,0,0,0]	(1,0,0,0,0,0,0,0,0,0,0)	2	1	1
(1,0,1)	[0,0,0,0,0,0,1,0]	(1,1,1,1,1,1,1,0,0,0,1)	2	8	1

(n_c, n_d, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(2,0,1)	[0,0,0,1,0,0,0,0]	(1,1,1,1,1,2,3,2,1,0,2)	2	15	1
(3,0,1)	[0,1,0,0,0,0,0,1]	(1,1,1,2,3,4,5,3,1,0,3)	2	24	1
(3,0,1)	[1,0,0,0,0,0,0,0]	(1,1,2,3,4,5,6,4,2,0,3)	0	31	1
(4,0,1)	[1,0,0,0,0,1,0,0]	(1,1,2,3,4,5,6,4,2,0,4)	2	32	1
(4,0,1)	[0,0,0,0,0,0,0,2]	(1,2,3,4,5,6,7,4,1,0,4)	2	37	1
(4,0,1)	[0,0,0,0,0,0,1,0]	(1,2,3,4,5,6,7,4,2,0,4)	0	38	1
(5,0,1)	[0,0,0,0,1,0,0,1]	(1,2,3,4,5,6,8,5,2,0,5)	2	41	1
(5,0,1)	[1,0,1,0,0,0,0,0]	(1,1,2,3,5,7,9,6,3,0,5)	2	42	1
(5,0,1)	[0,0,0,1,0,0,0,0]	(1,2,3,4,5,7,9,6,3,0,5)	0	45	1
(0,1,1)	[0,0,0,0,0,0,0,0]	(1,1,1,1,1,1,1,1,1,1,0)	2	10	1
(1,1,1)	[0,0,0,0,0,1,0,0]	(1,1,1,1,1,1,1,1,1,1,1)	2	11	1
(2,1,1)	[0,0,0,1,0,0,0,1]	(1,1,1,1,1,2,3,2,1,1,2)	2	16	1
(2,1,1)	[0,0,1,0,0,0,0,0]	(1,1,1,1,2,3,4,3,2,1,2)	0	21	2
(3,1,1)	[0,0,1,0,0,1,0,0]	(1,1,1,1,2,3,4,3,2,1,3)	2	22	1
(3,1,1)	[0,1,0,0,0,0,0,2]	(1,1,1,2,3,4,5,3,1,1,3)	2	25	1
(3,1,1)	[0,1,0,0,0,0,1,0]	(1,1,1,2,3,4,5,3,2,1,3)	0	26	2
(3,1,1)	[1,0,0,0,0,0,0,1]	(1,1,2,3,4,5,6,4,2,1,3)	−2	32	4
(3,1,1)	[0,0,0,0,0,0,0,0]	(1,2,3,4,5,6,7,5,3,1,3)	−4	40	3
(4,1,1)	[0,1,0,0,1,0,0,1]	(1,1,1,2,3,4,6,4,2,1,4)	2	29	1
(4,1,1)	[0,0,2,0,0,0,0,0]	(1,1,1,1,3,5,7,5,3,1,4)	2	32	1
(4,1,1)	[1,0,0,0,0,1,0,1]	(1,1,2,3,4,5,6,4,2,1,4)	0	33	3
(4,1,1)	[0,1,0,1,0,0,0,0]	(1,1,1,2,3,5,7,5,3,1,4)	0	33	2
(4,1,1)	[1,0,0,0,1,0,0,0]	(1,1,2,3,4,5,7,5,3,1,4)	−2	36	4
(4,1,1)	[0,0,0,0,0,0,0,3]	(1,2,3,4,5,6,7,4,1,1,4)	2	38	1
(4,1,1)	[0,0,0,0,0,0,1,1]	(1,2,3,4,5,6,7,4,2,1,4)	−2	39	4
(4,1,1)	[0,0,0,0,0,1,0,0]	(1,2,3,4,5,6,7,5,3,1,4)	−4	41	7
(2,2,1)	[0,0,1,0,0,0,0,1]	(1,1,1,1,2,3,4,3,2,2,2)	2	22	1
(2,2,1)	[0,1,0,0,0,0,0,0]	(1,1,1,2,3,4,5,4,3,2,2)	0	28	1
(3,2,1)	[0,0,1,0,1,0,0,0]	(1,1,1,1,2,3,5,4,3,2,3)	2	26	1
(3,2,1)	[0,1,0,0,0,0,1,1]	(1,1,1,2,3,4,5,3,2,2,3)	2	27	1
(3,2,1)	[0,1,0,0,0,1,0,0]	(1,1,1,2,3,4,5,4,3,2,3)	0	29	2
(3,2,1)	[1,0,0,0,0,0,0,2]	(1,1,2,3,4,5,6,4,2,2,3)	0	33	2
(3,2,1)	[1,0,0,0,0,0,1,0]	(1,1,2,3,4,5,6,4,3,2,3)	−2	34	4
(3,2,1)	[0,0,0,0,0,0,0,1]	(1,2,3,4,5,6,7,5,3,2,3)	−4	41	5
(4,2,1)	[0,1,0,0,1,0,1,0]	(1,1,1,2,3,4,6,4,3,2,4)	2	31	1
(4,2,1)	[0,0,2,0,0,0,0,1]	(1,1,1,1,3,5,7,5,3,2,4)	2	33	1
(4,2,1)	[1,0,0,0,0,1,0,2]	(1,1,2,3,4,5,6,4,2,2,4)	2	34	1
(4,2,1)	[0,1,0,1,0,0,0,1]	(1,1,1,2,3,5,7,5,3,2,4)	0	34	3
(4,2,1)	[1,0,0,0,0,1,1,0]	(1,1,2,3,4,5,6,4,3,2,4)	0	35	2
(4,2,1)	[1,0,0,0,1,0,0,1]	(1,1,2,3,4,5,7,5,3,2,4)	−2	37	7
(4,2,1)	[0,1,1,0,0,0,0,0]	(1,1,1,2,4,6,8,6,4,2,4)	−2	39	5

(n_c,n_d,n_0)	p_i	β	β^2	$ht(\beta)$	μ
(4,2,1)	[0,0,0,0,0,0,1,2]	(1,2,3,4,5,6,7,4,2,2,4)	0	40	2
(4,2,1)	[1,0,0,1,0,0,0,0]	(1,1,2,3,4,6,8,6,4,2,4)	−4	41	8
(4,2,1)	[0,0,0,0,0,0,2,0]	(1,2,3,4,5,6,7,4,3,2,4)	−2	41	3
(4,2,1)	[0,0,0,0,0,1,0,1]	(1,2,3,4,5,6,7,5,3,2,4)	−4	42	11
(4,2,1)	[0,0,0,0,1,0,0,0]	(1,2,3,4,5,6,8,6,4,2,4)	−6	45	13

B.3 E_6^{+++1}

(n_d,n_c,n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,0,0)	[0,0,0,0,0,0,0]	(0,0,0,0,0,0,0,0,0,1)	2	1	1
(0,1,0)	[0,0,0,0,1,0,0]	(0,0,0,0,0,0,0,0,1,0)	2	1	1
(1,1,0)	[0,0,0,0,1,0,0]	(0,0,0,0,0,0,0,0,1,1)	2	2	1
(0,2,0)	[0,1,0,0,0,0,0]	(0,0,0,1,2,3,2,1,2,0)	2	11	1
(1,2,0)	[0,0,1,0,0,0,1]	(0,0,0,0,1,2,1,0,2,1)	2	7	1
(1,2,0)	[0,1,0,0,0,0,0]	(0,0,0,1,2,3,2,1,2,1)	0	12	1
(2,2,0)	[0,1,0,0,0,0,0]	(0,0,0,1,2,3,2,1,2,2)	2	13	1
(0,3,0)	[0,0,0,0,0,0,1]	(0,1,2,3,4,5,3,1,3,0)	2	22	1
(1,3,0)	[0,1,0,0,1,0,0]	(0,0,0,1,2,3,2,1,3,1)	2	13	1
(1,3,0)	[1,0,0,0,0,0,2]	(0,0,1,2,3,4,2,0,3,1)	2	16	1
(1,3,0)	[1,0,0,0,0,1,0]	(0,0,1,2,3,4,2,1,3,1)	0	17	1
(1,3,0)	[0,0,0,0,0,0,1]	(0,1,2,3,4,5,3,1,3,1)	−2	23	2
(2,3,0)	[0,1,0,0,1,0,0]	(0,0,0,1,2,3,2,1,3,2)	2	14	1
(2,3,0)	[1,0,0,0,0,0,2]	(0,0,1,2,3,4,2,0,3,2)	2	17	1
(2,3,0)	[1,0,0,0,0,1,0]	(0,0,1,2,3,4,2,1,3,2)	0	18	1
(2,3,0)	[0,0,0,0,0,0,1]	(0,1,2,3,4,5,3,1,3,2)	−2	24	2
(3,3,0)	[0,0,0,0,0,0,1]	(0,1,2,3,4,5,3,1,3,3)	2	25	1
(1,4,0)	[1,0,0,1,0,0,1]	(0,0,1,2,3,5,3,1,4,1)	2	20	1
(1,4,0)	[0,2,0,0,0,0,0]	(0,0,0,2,4,6,4,2,4,1)	2	23	1
(1,4,0)	[1,0,1,0,0,0,0]	(0,0,1,2,4,6,4,2,4,1)	0	24	1
(1,4,0)	[0,0,0,0,1,0,1]	(0,1,2,3,4,5,3,1,4,1)	0	24	2
(1,4,0)	[0,0,0,1,0,0,0]	(0,1,2,3,4,6,4,2,4,1)	−2	27	2
(2,4,0)	[1,0,0,0,1,1,0]	(0,0,1,2,3,4,2,1,4,2)	2	19	1
(2,4,0)	[0,1,1,0,0,0,1]	(0,0,0,1,3,5,3,1,4,2)	2	19	1
(2,4,0)	[1,0,0,1,0,0,1]	(0,0,1,2,3,5,3,1,4,2)	0	21	2
(2,4,0)	[0,0,0,0,0,1,2]	(0,1,2,3,4,5,2,0,4,2)	2	23	1
(2,4,0)	[0,2,0,0,0,0,0]	(0,0,0,2,4,6,4,2,4,2)	0	24	1
(2,4,0)	[0,0,0,0,0,2,0]	(0,1,2,3,4,5,2,1,4,2)	0	24	0
(2,4,0)	[1,0,1,0,0,0,0]	(0,0,1,2,4,6,4,2,4,2)	−2	25	3
(2,4,0)	[0,0,0,0,1,0,1]	(0,1,2,3,4,5,3,1,4,2)	−2	25	4
(2,4,0)	[0,0,0,1,0,0,0]	(0,1,2,3,4,6,4,2,4,2)	−4	28	3
(3,4,0)	[1,0,0,1,0,0,1]	(0,0,1,2,3,5,3,1,4,3)	2	22	1
(3,4,0)	[0,2,0,0,0,0,0]	(0,0,0,2,4,6,4,2,4,3)	2	25	1

(n_d, n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(3,4,0)	[1,0,1,0,0,0,0]	(0,0,1,2,4,6,4,2,4,3)	0	26	1
(3,4,0)	[0,0,0,0,1,0,1]	(0,1,2,3,4,5,3,1,4,3)	0	26	2
(3,4,0)	[0,0,0,1,0,0,0]	(0,1,2,3,4,6,4,2,4,3)	-2	29	2
(1,5,0)	[1,1,0,0,0,1,0]	(0,0,1,3,5,7,4,2,5,1)	2	28	1
(1,5,0)	[0,0,0,1,1,0,0]	(0,1,2,3,4,6,4,2,5,1)	2	28	1
(1,5,0)	[0,0,1,0,0,0,2]	(0,1,2,3,5,7,4,1,5,1)	2	29	1
(1,5,0)	[0,0,1,0,0,1,0]	(0,1,2,3,5,7,4,2,5,1)	0	30	1
(1,5,0)	[2,0,0,0,0,0,1]	(0,0,2,4,6,8,5,2,5,1)	0	33	1
(1,5,0)	[0,1,0,0,0,0,1]	(0,1,2,4,6,8,5,2,5,1)	-2	34	4
(1,5,0)	[1,0,0,0,0,0,0]	(0,1,3,5,7,9,6,3,5,1)	-4	40	2
(2,5,0)	[1,0,1,0,0,1,1]	(0,0,1,2,4,6,3,1,5,2)	2	24	1
(2,5,0)	[1,0,0,2,0,0,0]	(0,0,1,2,3,6,4,2,5,2)	2	25	1
(2,5,0)	[0,2,0,0,1,0,0]	(0,0,0,2,4,6,4,2,5,2)	2	25	1
(2,5,0)	[1,0,1,0,1,0,0]	(0,0,1,2,4,6,4,2,5,2)	0	26	2
(2,5,0)	[0,0,0,0,2,0,1]	(0,1,2,3,4,5,3,1,5,2)	2	26	1
(2,5,0)	[0,0,0,1,0,1,1]	(0,1,2,3,4,6,3,1,5,2)	0	27	2
(2,5,0)	[1,1,0,0,0,0,2]	(0,0,1,3,5,7,4,1,5,2)	0	28	2
(2,5,0)	[1,1,0,0,0,1,0]	(0,0,1,3,5,7,4,2,5,2)	-2	29	5
(2,5,0)	[0,0,0,1,1,0,0]	(0,1,2,3,4,6,4,2,5,2)	-2	29	4
(2,5,0)	[0,0,1,0,0,0,2]	(0,1,2,3,5,7,4,1,5,2)	-2	30	5
(2,5,0)	[0,0,1,0,0,1,0]	(0,1,2,3,5,7,4,2,5,2)	-4	31	7
(2,5,0)	[2,0,0,0,0,0,1]	(0,0,2,4,6,8,5,2,5,2)	-4	34	4
(2,5,0)	[0,1,0,0,0,0,1]	(0,1,2,4,6,8,5,2,5,2)	-6	35	13
(2,5,0)	[1,0,0,0,0,0,0]	(0,1,3,5,7,9,6,3,5,2)	-8	41	6
(3,5,0)	[1,0,1,0,0,1,1]	(0,0,1,2,4,6,3,1,5,3)	2	25	1
(3,5,0)	[1,0,0,2,0,0,0]	(0,0,1,2,3,6,4,2,5,3)	2	26	1
(3,5,0)	[0,2,0,0,1,0,0]	(0,0,0,2,4,6,4,2,5,3)	2	26	1
(3,5,0)	[1,0,1,0,1,0,0]	(0,0,1,2,4,6,4,2,5,3)	0	27	2
(3,5,0)	[0,0,0,0,2,0,1]	(0,1,2,3,4,5,3,1,5,3)	2	27	1
(3,5,0)	[0,0,0,1,0,1,1]	(0,1,2,3,4,6,3,1,5,3)	0	28	2
(3,5,0)	[1,1,0,0,0,0,2]	(0,0,1,3,5,7,4,1,5,3)	0	29	2
(3,5,0)	[1,1,0,0,0,1,0]	(0,0,1,3,5,7,4,2,5,3)	-2	30	5
(3,5,0)	[0,0,0,1,1,0,0]	(0,1,2,3,4,6,4,2,5,3)	-2	30	4
(3,5,0)	[0,0,1,0,0,0,2]	(0,1,2,3,5,7,4,1,5,3)	-2	31	5
(3,5,0)	[0,0,1,0,0,1,0]	(0,1,2,3,5,7,4,2,5,3)	-4	32	7
(3,5,0)	[2,0,0,0,0,0,1]	(0,0,2,4,6,8,5,2,5,3)	-4	35	4
(3,5,0)	[0,1,0,0,0,0,1]	(0,1,2,4,6,8,5,2,5,3)	-6	36	13
(3,5,0)	[1,0,0,0,0,0,0]	(0,1,3,5,7,9,6,3,5,3)	-8	42	6
(4,5,0)	[1,1,0,0,0,1,0]	(0,0,1,3,5,7,4,2,5,4)	2	31	1
(4,5,0)	[0,0,0,1,1,0,0]	(0,1,2,3,4,6,4,2,5,4)	2	31	1
(4,5,0)	[0,0,1,0,0,0,2]	(0,1,2,3,5,7,4,1,5,4)	2	32	1

(n_d,n_c,n_0)	p_i	β	β^2	$ht(\beta)$	μ
(4,5,0)	[0,0,1,0,0,1,0]	(0,1,2,3,5,7,4,2,5,4)	0	33	1
(4,5,0)	[2,0,0,0,0,0,1]	(0,0,2,4,6,8,5,2,5,4)	0	36	1
(4,5,0)	[0,1,0,0,0,0,1]	(0,1,2,4,6,8,5,2,5,4)	-2	37	4
(4,5,0)	[1,0,0,0,0,0,0]	(0,1,3,5,7,9,6,3,5,4)	-4	43	2
(0,0,1)	[1,0,0,0,0,0,0]	(1,0,0,0,0,0,0,0,0,0)	2	1	1
(0,1,1)	[0,0,0,0,0,1,0]	(1,1,1,1,1,1,0,0,1,0)	2	7	1
(1,1,1)	[0,0,0,0,0,1,0]	(1,1,1,1,1,1,0,0,1,1)	2	8	1
(0,2,1)	[0,0,1,0,0,0,0]	(1,1,1,1,2,3,2,1,2,0)	2	14	1
(1,2,1)	[0,0,0,1,0,0,1]	(1,1,1,1,1,2,1,0,2,1)	2	11	1
(1,2,1)	[0,0,1,0,0,0,0]	(1,1,1,1,2,3,2,1,2,1)	0	15	2
(2,2,1)	[0,0,1,0,0,0,0]	(1,1,1,1,2,3,2,1,2,2)	2	16	1
(0,3,1)	[1,0,0,0,0,0,1]	(1,1,2,3,4,5,3,1,3,0)	2	23	1
(0,3,1)	[0,0,0,0,0,0,0]	(1,2,3,4,5,6,4,2,3,0)	0	30	1
(1,3,1)	[0,0,1,0,1,0,0]	(1,1,1,1,2,3,2,1,3,1)	2	16	1
(1,3,1)	[0,1,0,0,0,0,2]	(1,1,1,2,3,4,2,0,3,1)	2	18	1
(1,3,1)	[0,1,0,0,0,1,0]	(1,1,1,2,3,4,2,1,3,1)	0	19	2
(1,3,1)	[1,0,0,0,0,0,1]	(1,1,2,3,4,5,3,1,3,1)	-2	24	4
(1,3,1)	[0,0,0,0,0,0,0]	(1,2,3,4,5,6,4,2,3,1)	-4	31	3
(2,3,1)	[0,0,1,0,1,0,0]	(1,1,1,1,2,3,2,1,3,2)	2	17	1
(2,3,1)	[0,1,0,0,0,0,2]	(1,1,1,2,3,4,2,0,3,2)	2	19	1
(2,3,1)	[0,1,0,0,0,1,0]	(1,1,1,2,3,4,2,1,3,2)	0	20	2
(2,3,1)	[1,0,0,0,0,0,1]	(1,1,2,3,4,5,3,1,3,2)	-2	25	4
(2,3,1)	[0,0,0,0,0,0,0]	(1,2,3,4,5,6,4,2,3,2)	-4	32	3
(3,3,1)	[1,0,0,0,0,0,1]	(1,1,2,3,4,5,3,1,3,3)	2	26	1
(3,3,1)	[0,0,0,0,0,0,0]	(1,2,3,4,5,6,4,2,3,3)	0	33	1
(0,4,1)	[0,0,0,0,1,0,0]	(1,2,3,4,5,6,4,2,4,0)	2	31	1
(1,4,1)	[0,1,0,1,0,0,1]	(1,1,1,2,3,5,3,1,4,1)	2	22	1
(1,4,1)	[1,0,0,0,1,0,1]	(1,1,2,3,4,5,3,1,4,1)	0	25	3
(1,4,1)	[0,1,1,0,0,0,0]	(1,1,1,2,4,6,4,2,4,1)	0	26	2
(1,4,1)	[1,0,0,1,0,0,0]	(1,1,2,3,4,6,4,2,4,1)	-2	28	4
(1,4,1)	[0,0,0,0,0,0,3]	(1,2,3,4,5,6,3,0,4,1)	2	29	1
(1,4,1)	[0,0,0,0,0,1,1]	(1,2,3,4,5,6,3,1,4,1)	-2	30	4
(1,4,1)	[0,0,0,0,1,0,0]	(1,2,3,4,5,6,4,2,4,1)	-4	32	7
(2,4,1)	[0,1,0,0,1,1,0]	(1,1,1,2,3,4,2,1,4,2)	2	21	1
(2,4,1)	[0,0,2,0,0,0,1]	(1,1,1,1,3,5,3,1,4,2)	2	22	1
(2,4,1)	[0,1,0,1,0,0,1]	(1,1,1,2,3,5,3,1,4,2)	0	23	3
(2,4,1)	[1,0,0,0,0,1,2]	(1,1,2,3,4,5,2,0,4,2)	2	24	1
(2,4,1)	[1,0,0,0,0,2,0]	(1,1,2,3,4,5,2,1,4,2)	0	25	1
(2,4,1)	[1,0,0,0,1,0,1]	(1,1,2,3,4,5,3,1,4,2)	-2	26	7
(2,4,1)	[0,1,1,0,0,0,0]	(1,1,1,2,4,6,4,2,4,2)	-2	27	5
(2,4,1)	[1,0,0,1,0,0,0]	(1,1,2,3,4,6,4,2,4,2)	-4	29	8

(n_d, n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(2,4,1)	[0,0,0,0,0,0,3]	(1,2,3,4,5,6,3,0,4,2)	0	30	2
(2,4,1)	[0,0,0,0,0,1,1]	(1,2,3,4,5,6,3,1,4,2)	-4	31	9
(2,4,1)	[0,0,0,0,1,0,0]	(1,2,3,4,5,6,4,2,4,2)	-6	33	13
(3,4,1)	[0,1,0,1,0,0,1]	(1,1,1,2,3,5,3,1,4,3)	2	24	1
(3,4,1)	[1,0,0,0,1,0,1]	(1,1,2,3,4,5,3,1,4,3)	0	27	3
(3,4,1)	[0,1,1,0,0,0,0]	(1,1,1,2,4,6,4,2,4,3)	0	28	2
(3,4,1)	[1,0,0,1,0,0,0]	(1,1,2,3,4,6,4,2,4,3)	-2	30	4
(3,4,1)	[0,0,0,0,0,0,3]	(1,2,3,4,5,6,3,0,4,3)	2	31	1
(3,4,1)	[0,0,0,0,0,1,1]	(1,2,3,4,5,6,3,1,4,3)	-2	32	4
(3,4,1)	[0,0,0,0,1,0,0]	(1,2,3,4,5,6,4,2,4,3)	-4	34	7
(4,4,1)	[0,0,0,0,1,0,0]	(1,2,3,4,5,6,4,2,4,4)	2	35	1

B.4 G_2^{+++1}

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,0)	[0,0,0,1]	(0,0,0,0,0,1)	2/3	1	1
(2,0)	[0,0,1,0]	(0,0,0,0,1,2)	2/3	3	1
(3,0)	[0,0,1,1]	(0,0,0,0,1,3)	2	4	1
(3,0)	[0,1,0,0]	(0,0,0,1,2,3)	0	6	0
(4,0)	[0,1,0,1]	(0,0,0,1,2,4)	2/3	7	1
(4,0)	[1,0,0,0]	(0,0,1,2,3,4)	$-4/3$	10	0
(5,0)	[0,1,1,0]	(0,0,0,1,3,5)	2/3	9	1
(5,0)	[1,0,0,1]	(0,0,1,2,3,5)	$-4/3$	11	1
(5,0)	[0,0,0,0]	(0,1,2,3,4,5)	$-10/3$	15	0
(6,0)	[0,1,1,1]	(0,0,0,1,3,6)	2	10	1
(6,0)	[1,0,0,2]	(0,0,1,2,3,6)	0	12	0
(6,0)	[0,2,0,0]	(0,0,0,2,4,6)	0	12	0
(6,0)	[1,0,1,0]	(0,0,1,2,4,6)	-2	13	2
(6,0)	[0,0,0,1]	(0,1,2,3,4,6)	-4	16	1
(7,0)	[0,2,0,1]	(0,0,0,2,4,7)	2/3	13	1
(7,0)	[1,0,1,1]	(0,0,1,2,4,7)	$-4/3$	14	2
(7,0)	[1,1,0,0]	(0,0,1,3,5,7)	$-10/3$	16	2
(7,0)	[0,0,0,2]	(0,1,2,3,4,7)	$-10/3$	17	1
(7,0)	[0,0,1,0]	(0,1,2,3,5,7)	$-16/3$	18	2
(0,1)	[1,0,0,0]	(1,0,0,0,0,0)	2	1	1
(1,1)	[0,0,0,0]	(1,1,1,1,1,1)	2/3	6	1
(2,1)	[0,0,0,1]	(1,1,1,1,1,2)	2/3	7	1
(3,1)	[0,0,0,2]	(1,1,1,1,1,3)	2	8	1
(3,1)	[0,0,1,0]	(1,1,1,1,2,3)	0	9	1
(4,1)	[0,0,1,1]	(1,1,1,1,2,4)	2/3	10	1
(4,1)	[0,1,0,0]	(1,1,1,2,3,4)	$-4/3$	12	1
(5,1)	[0,0,2,0]	(1,1,1,1,3,5)	2/3	12	1

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(5,1)	[0,1,0,1]	(1,1,1,2,3,5)	$-4/3$	13	2
(5,1)	[1,0,0,0]	(1,1,2,3,4,5)	$-10/3$	16	1
(6,1)	[0,0,2,1]	(1,1,1,1,3,6)	2	13	1
(6,1)	[0,1,0,2]	(1,1,1,2,3,6)	0	14	1
(6,1)	[0,1,1,0]	(1,1,1,2,4,6)	-2	15	3
(6,1)	[1,0,0,1]	(1,1,2,3,4,6)	-4	17	3
(6,1)	[0,0,0,0]	(1,2,3,4,5,6)	-6	21	1
(7,1)	[0,1,1,1]	(1,1,1,2,4,7)	$-4/3$	16	3
(7,1)	[1,0,0,2]	(1,1,2,3,4,7)	$-10/3$	18	3
(7,1)	[0,2,0,0]	(1,1,1,3,5,7)	$-10/3$	18	3
(7,1)	[1,0,1,0]	(1,1,2,3,5,7)	$-16/3$	19	6
(7,1)	[0,0,0,1]	(1,2,3,4,5,7)	$-22/3$	22	4

B.5 F_4^{+++1}

(n_c, n_d, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,0,0)	[0,0,0,0,0]	(0,0,0,0,0,0,0,1)	1	1	1
(0,1,0)	[0,0,0,0,1]	(0,0,0,0,0,0,1,0)	1	1	1
(1,1,0)	[0,0,0,0,1]	(0,0,0,0,0,0,1,1)	1	2	1
(0,2,0)	[0,0,0,1,0]	(0,0,0,0,0,1,2,0)	2	3	1
(1,2,0)	[0,0,0,1,0]	(0,0,0,0,0,1,2,1)	1	4	1
(2,2,0)	[0,0,0,1,0]	(0,0,0,0,0,1,2,2)	2	5	1
(1,3,0)	[0,0,1,0,0]	(0,0,0,0,1,2,3,1)	1	7	1
(2,3,0)	[0,0,1,0,0]	(0,0,0,0,1,2,3,2)	1	8	1
(1,4,0)	[0,1,0,0,0]	(0,0,0,1,2,3,4,1)	1	11	1
(2,4,0)	[0,0,1,0,1]	(0,0,0,0,1,2,4,2)	2	9	1
(2,4,0)	[0,1,0,0,0]	(0,0,0,1,2,3,4,2)	0	12	1
(3,4,0)	[0,1,0,0,0]	(0,0,0,1,2,3,4,3)	1	13	1
(1,5,0)	[1,0,0,0,0]	(0,0,1,2,3,4,5,1)	1	16	1
(2,5,0)	[0,1,0,0,1]	(0,0,0,1,2,3,5,2)	1	13	1
(2,5,0)	[1,0,0,0,0]	(0,0,1,2,3,4,5,2)	-1	17	2
(3,5,0)	[0,1,0,0,1]	(0,0,0,1,2,3,5,3)	1	14	1
(3,5,0)	[1,0,0,0,0]	(0,0,1,2,3,4,5,3)	-1	18	2
(4,5,0)	[1,0,0,0,0]	(0,0,1,2,3,4,5,4)	1	19	1
(1,6,0)	[0,0,0,0,0]	(0,1,2,3,4,5,6,1)	1	22	1
(2,6,0)	[0,1,0,1,0]	(0,0,0,1,2,4,6,2)	2	15	1
(2,6,0)	[1,0,0,0,1]	(0,0,1,2,3,4,6,2)	0	18	1
(2,6,0)	[0,0,0,0,0]	(0,1,2,3,4,5,6,2)	-2	23	3
(3,6,0)	[0,1,0,1,0]	(0,0,0,1,2,4,6,3)	1	16	1
(3,6,0)	[1,0,0,0,1]	(0,0,1,2,3,4,6,3)	-1	19	3
(3,6,0)	[0,0,0,0,0]	(0,1,2,3,4,5,6,3)	-3	24	3
(4,6,0)	[0,1,0,1,0]	(0,0,0,1,2,4,6,4)	2	17	1

JHEP02(2004)033

(n_c, n_d, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(4,6,0)	[1,0,0,0,1]	(0,0,1,2,3,4,6,4)	0	20	1
(4,6,0)	[0,0,0,0,0]	(0,1,2,3,4,5,6,4)	−2	25	3
(5,6,0)	[0,0,0,0,0]	(0,1,2,3,4,5,6,5)	1	26	1
(0,0,1)	[1,0,0,0,0]	(1,0,0,0,0,0,0,0)	2	1	1
(0,1,1)	[0,0,0,0,0]	(1,1,1,1,1,1,1,0)	1	7	1
(1,1,1)	[0,0,0,0,0]	(1,1,1,1,1,1,1,1)	1	8	1
(0,2,1)	[0,0,0,0,1]	(1,1,1,1,1,1,2,0)	2	8	1
(1,2,1)	[0,0,0,0,1]	(1,1,1,1,1,1,2,1)	1	9	1
(2,2,1)	[0,0,0,0,1]	(1,1,1,1,1,1,2,2)	2	10	1
(1,3,1)	[0,0,0,1,0]	(1,1,1,1,1,2,3,1)	1	11	1
(2,3,1)	[0,0,0,1,0]	(1,1,1,1,1,2,3,2)	1	12	1
(1,4,1)	[0,0,1,0,0]	(1,1,1,1,2,3,4,1)	1	14	1
(2,4,1)	[0,0,0,1,1]	(1,1,1,1,1,2,4,2)	2	13	1
(2,4,1)	[0,0,1,0,0]	(1,1,1,1,2,3,4,2)	0	15	2
(3,4,1)	[0,0,1,0,0]	(1,1,1,1,2,3,4,3)	1	16	1
(1,5,1)	[0,1,0,0,0]	(1,1,1,2,3,4,5,1)	1	18	1
(2,5,1)	[0,0,1,0,1]	(1,1,1,1,2,3,5,2)	1	16	1
(2,5,1)	[0,1,0,0,0]	(1,1,1,2,3,4,5,2)	−1	19	3
(3,5,1)	[0,0,1,0,1]	(1,1,1,1,2,3,5,3)	1	17	1
(3,5,1)	[0,1,0,0,0]	(1,1,1,2,3,4,5,3)	−1	20	3
(4,5,1)	[0,1,0,0,0]	(1,1,1,2,3,4,5,4)	1	21	1
(1,6,1)	[1,0,0,0,0]	(1,1,2,3,4,5,6,1)	1	23	1
(2,6,1)	[0,0,1,1,0]	(1,1,1,1,2,4,6,2)	2	18	1
(2,6,1)	[0,1,0,0,1]	(1,1,1,2,3,4,6,2)	0	20	2
(2,6,1)	[1,0,0,0,0]	(1,1,2,3,4,5,6,2)	−2	24	4
(3,6,1)	[0,0,1,1,0]	(1,1,1,1,2,4,6,3)	1	19	1
(3,6,1)	[0,1,0,0,1]	(1,1,1,2,3,4,6,3)	−1	21	4
(3,6,1)	[1,0,0,0,0]	(1,1,2,3,4,5,6,3)	−3	25	6
(4,6,1)	[0,0,1,1,0]	(1,1,1,1,2,4,6,4)	2	20	1
(4,6,1)	[0,1,0,0,1]	(1,1,1,2,3,4,6,4)	0	22	2
(4,6,1)	[1,0,0,0,0]	(1,1,2,3,4,5,6,4)	−2	26	4
(5,6,1)	[1,0,0,0,0]	(1,1,2,3,4,5,6,5)	1	27	1

B.6 D_8^{+++1}

(n_c, n_d, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,0,0)	[0,0,0,1,0,0,0,0,0]	(0,0,0,0,0,0,0,0,0,0,0,1)	2	1	1
(2,0,0)	[1,0,0,0,0,0,1,0,0]	(0,0,1,2,3,2,1,0,0,0,0,2)	2	11	1
(2,0,0)	[0,0,0,0,0,0,0,1,0]	(0,1,2,3,4,3,2,1,0,0,0,2)	0	18	0
(0,1,0)	[0,0,0,0,0,0,0,1,0]	(0,0,0,0,0,0,0,0,0,0,1,0)	2	1	1
(1,1,0)	[0,0,1,0,0,0,0,0,1]	(0,0,0,0,1,1,1,1,1,0,1,1)	2	7	1
(1,1,0)	[0,1,0,0,0,0,0,0,0]	(0,0,0,1,2,2,2,2,2,1,1,1)	0	14	1

(n_c,n_d,n_0)	p_i	β	β^2	$ht(\beta)$	μ
(2,1,0)	[1,0,0,0,0,1,0,0,1]	(0,0,1,2,3,2,1,1,1,0,1,2)	2	14	1
(2,1,0)	[0,1,0,1,0,0,0,0,0]	(0,0,0,1,2,2,2,2,2,1,1,2)	2	15	1
(2,1,0)	[1,0,0,0,1,0,0,0,0]	(0,0,1,2,3,2,2,2,2,1,1,2)	0	18	1
(2,1,0)	[0,0,0,0,0,0,1,0,1]	(0,1,2,3,4,3,2,1,1,0,1,2)	0	20	1
(2,1,0)	[0,0,0,0,0,1,0,0,0]	(0,1,2,3,4,3,2,2,2,1,1,2)	-2	23	2
(1,2,0)	[0,1,0,0,0,0,0,1,0]	(0,0,0,1,2,2,2,2,2,1,2,1)	2	15	1
(1,2,0)	[1,0,0,0,0,0,0,0,1]	(0,0,1,2,3,3,3,3,3,1,2,1)	0	22	1
(1,2,0)	[0,0,0,0,0,0,0,0,0]	(0,1,2,3,4,4,4,4,4,2,2,1)	-2	31	1
(2,2,0)	[1,0,0,0,1,0,0,1,0]	(0,0,1,2,3,2,2,2,2,1,2,2)	2	19	1
(2,2,0)	[0,1,1,0,0,0,0,0,1]	(0,0,0,1,3,3,3,3,3,1,2,2)	2	21	1
(2,2,0)	[1,0,0,1,0,0,0,0,1]	(0,0,1,2,3,3,3,3,3,1,2,2)	0	23	2
(2,2,0)	[0,0,0,0,0,1,0,0,2]	(0,1,2,3,4,3,2,2,2,0,2,2)	2	23	1
(2,2,0)	[0,0,0,0,0,1,0,1,0]	(0,1,2,3,4,3,2,2,2,1,2,2)	0	24	1
(2,2,0)	[0,0,0,0,1,0,0,0,1]	(0,1,2,3,4,3,3,3,3,1,2,2)	-2	27	3
(2,2,0)	[0,2,0,0,0,0,0,0,0]	(0,0,0,2,4,4,4,4,4,2,2,2)	0	28	1
(2,2,0)	[1,0,1,0,0,0,0,0,0]	(0,0,1,2,4,4,4,4,4,2,2,2)	-2	29	3
(2,2,0)	[0,0,0,1,0,0,0,0,0]	(0,1,2,3,4,4,4,4,4,2,2,2)	-4	32	3
(1,3,0)	[1,0,0,0,0,0,1,0,0]	(0,0,1,2,3,3,3,3,4,2,3,1)	2	25	1
(1,3,0)	[0,0,0,0,0,0,0,1,0]	(0,1,2,3,4,4,4,4,4,2,3,1)	0	32	1
(2,3,0)	[1,0,0,1,0,0,1,0,0]	(0,0,1,2,3,3,3,3,4,2,3,2)	2	26	1
(2,3,0)	[0,0,0,0,1,0,0,1,1]	(0,1,2,3,4,3,3,3,3,1,3,2)	2	28	1
(2,3,0)	[1,0,1,0,0,0,0,0,2]	(0,0,1,2,4,4,4,4,4,1,3,2)	2	29	1
(2,3,0)	[0,2,0,0,0,0,0,1,0]	(0,0,0,2,4,4,4,4,4,2,3,2)	2	29	1
(2,3,0)	[1,0,1,0,0,0,0,1,0]	(0,0,1,2,4,4,4,4,4,2,3,2)	0	30	2
(2,3,0)	[0,0,0,0,1,0,1,0,0]	(0,1,2,3,4,3,3,3,4,2,3,2)	0	30	1
(2,3,0)	[0,0,0,1,0,0,0,0,2]	(0,1,2,3,4,4,4,4,4,1,3,2)	0	32	1
(2,3,0)	[0,0,0,1,0,0,0,1,0]	(0,1,2,3,4,4,4,4,4,2,3,2)	-2	33	4
(2,3,0)	[1,1,0,0,0,0,0,0,1]	(0,0,1,3,5,5,5,5,5,2,3,2)	-2	36	4
(2,3,0)	[0,0,1,0,0,0,0,0,1]	(0,1,2,3,5,5,5,5,5,2,3,2)	-4	38	6
(2,3,0)	[2,0,0,0,0,0,0,0,0]	(0,0,2,4,6,6,6,6,6,3,3,2)	-4	44	2
(2,3,0)	[0,1,0,0,0,0,0,0,0]	(0,1,2,4,6,6,6,6,6,3,3,2)	-6	45	6
(0,0,1)	[1,0,0,0,0,0,0,0,0]	(1,0,0,0,0,0,0,0,0,0,0,0)	2	1	1
(1,0,1)	[0,0,0,0,1,0,0,0,0]	(1,1,1,1,1,0,0,0,0,0,0,1)	2	6	1
(2,0,1)	[0,1,0,0,0,0,1,0,0]	(1,1,1,2,3,2,1,0,0,0,0,2)	2	13	1
(2,0,1)	[1,0,0,0,0,0,0,1,0]	(1,1,2,3,4,3,2,1,0,0,0,2)	0	19	1
(2,0,1)	[0,0,0,0,0,0,0,0,1]	(1,2,3,4,5,4,3,2,1,0,0,2)	-2	27	1
(0,1,1)	[0,0,0,0,0,0,0,0,1]	(1,1,1,1,1,1,1,1,1,0,1,0)	2	10	1
(1,1,1)	[0,0,0,1,0,0,0,0,1]	(1,1,1,1,1,1,1,1,1,0,1,1)	2	11	1
(1,1,1)	[0,0,1,0,0,0,0,0,0]	(1,1,1,1,2,2,2,2,2,1,1,1)	0	17	2
(2,1,1)	[0,1,0,0,0,1,0,0,1]	(1,1,1,2,3,2,1,1,1,0,1,2)	2	16	1
(2,1,1)	[0,0,1,1,0,0,0,0,0]	(1,1,1,1,2,2,2,2,2,1,1,2)	2	18	1

(n_c, n_d, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(2,1,1)	[0,1,0,0,1,0,0,0,0]	(1,1,1,2,3,2,2,2,2,1,1,2)	0	20	2
(2,1,1)	[1,0,0,0,0,0,1,0,1]	(1,1,2,3,4,3,2,1,1,0,1,2)	0	21	2
(2,1,1)	[1,0,0,0,0,1,0,0,0]	(1,1,2,3,4,3,2,2,2,1,1,2)	-2	24	4
(2,1,1)	[0,0,0,0,0,0,0,1,1]	(1,2,3,4,5,4,3,2,1,0,1,2)	-2	28	2
(2,1,1)	[0,0,0,0,0,0,1,0,0]	(1,2,3,4,5,4,3,2,2,1,1,2)	-4	30	5
(1,2,1)	[0,0,1,0,0,0,0,1,0]	(1,1,1,1,2,2,2,2,2,1,2,1)	2	18	1
(1,2,1)	[0,1,0,0,0,0,0,0,1]	(1,1,1,2,3,3,3,3,3,1,2,1)	0	24	2
(1,2,1)	[1,0,0,0,0,0,0,0,0]	(1,1,2,3,4,4,4,4,4,2,2,1)	-2	32	2
(2,2,1)	[0,1,0,0,1,0,0,1,0]	(1,1,1,2,3,2,2,2,2,1,2,2)	2	21	1
(2,2,1)	[1,0,0,0,0,1,0,0,2]	(1,1,2,3,4,3,2,2,2,0,2,2)	2	24	1
(2,2,1)	[0,0,2,0,0,0,0,0,1]	(1,1,1,1,3,3,3,3,3,1,2,2)	2	24	1
(2,2,1)	[1,0,0,0,0,1,0,1,0]	(1,1,2,3,4,3,2,2,2,1,2,2)	0	25	2
(2,2,1)	[0,1,0,1,0,0,0,0,1]	(1,1,1,2,3,3,3,3,3,1,2,2)	0	25	3
(2,2,1)	[1,0,0,0,1,0,0,0,1]	(1,1,2,3,4,3,3,3,3,1,2,2)	-2	28	6
(2,2,1)	[0,0,0,0,0,0,0,2,1]	(1,2,3,4,5,4,3,2,1,0,2,2)	2	29	1
(2,2,1)	[0,0,0,0,0,0,1,0,2]	(1,2,3,4,5,4,3,2,2,0,2,2)	0	30	0
(2,2,1)	[0,1,1,0,0,0,0,0,0]	(1,1,1,2,4,4,4,4,4,2,2,2)	-2	31	5
(2,2,1)	[0,0,0,0,0,0,1,1,0]	(1,2,3,4,5,4,3,2,2,1,2,2)	-2	31	2
(2,2,1)	[1,0,0,1,0,0,0,0,0]	(1,1,2,3,4,4,4,4,4,2,2,2)	-4	33	8
(2,2,1)	[0,0,0,0,0,1,0,0,1]	(1,2,3,4,5,4,3,3,3,1,2,2)	-4	33	11
(2,2,1)	[0,0,0,0,1,0,0,0,0]	(1,2,3,4,5,4,4,4,4,2,2,2)	-6	37	9
(1,3,1)	[0,1,0,0,0,0,1,0,0]	(1,1,1,2,3,3,3,3,4,2,3,1)	2	27	1
(1,3,1)	[1,0,0,0,0,0,0,1,0]	(1,1,2,3,4,4,4,4,4,2,3,1)	0	33	2
(1,3,1)	[0,0,0,0,0,0,0,0,1]	(1,2,3,4,5,5,5,5,5,2,3,1)	-2	41	2
(2,3,1)	[0,1,0,1,0,0,1,0,0]	(1,1,1,2,3,3,3,3,4,2,3,2)	2	28	1
(2,3,1)	[1,0,0,0,1,0,0,1,1]	(1,1,2,3,4,3,3,3,3,1,3,2)	2	29	1
(2,3,1)	[1,0,0,0,1,0,1,0,0]	(1,1,2,3,4,3,3,3,4,2,3,2)	0	31	2
(2,3,1)	[0,1,1,0,0,0,0,0,2]	(1,1,1,2,4,4,4,4,4,1,3,2)	2	31	1
(2,3,1)	[0,1,1,0,0,0,0,1,0]	(1,1,1,2,4,4,4,4,4,2,3,2)	0	32	3
(2,3,1)	[1,0,0,1,0,0,0,0,2]	(1,1,2,3,4,4,4,4,4,1,3,2)	0	33	2
(2,3,1)	[1,0,0,1,0,0,0,1,0]	(1,1,2,3,4,4,4,4,4,2,3,2)	-2	34	7
(2,3,1)	[0,0,0,0,0,1,0,1,1]	(1,2,3,4,5,4,3,3,3,1,3,2)	0	34	2
(2,3,1)	[0,0,0,0,0,1,1,0,0]	(1,2,3,4,5,4,3,3,4,2,3,2)	-2	36	3
(2,3,1)	[0,0,0,0,1,0,0,0,2]	(1,2,3,4,5,4,4,4,4,1,3,2)	-2	37	4
(2,3,1)	[0,2,0,0,0,0,0,0,1]	(1,1,1,3,5,5,5,5,5,2,3,2)	-2	38	5
(2,3,1)	[0,0,0,0,1,0,0,1,0]	(1,2,3,4,5,4,4,4,4,2,3,2)	-4	38	11
(2,3,1)	[1,0,1,0,0,0,0,0,1]	(1,1,2,3,5,5,5,5,5,2,3,2)	-4	39	13
(2,3,1)	[0,0,0,1,0,0,0,0,1]	(1,2,3,4,5,5,5,5,5,2,3,2)	-6	42	19
(2,3,1)	[1,1,0,0,0,0,0,0,0]	(1,1,2,4,6,6,6,6,6,3,3,2)	-6	46	12
(2,3,1)	[0,0,1,0,0,0,0,0,0]	(1,2,3,4,6,6,6,6,6,3,3,2)	-8	48	19

JHEP02(2004)033

B.7 B_8^{+++1}

(n_c, n_d, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,0,0)	[0,0,0,1,0,0,0,0,0]	(0,0,0,0,0,0,0,0,0,0,1)	2	1	1
(2,0,0)	[1,0,0,0,0,0,1,0,0]	(0,0,1,2,3,2,1,0,0,0,2)	2	11	1
(2,0,0)	[0,0,0,0,0,0,0,1,0]	(0,1,2,3,4,3,2,1,0,0,2)	0	18	0
(3,0,0)	[0,0,0,1,0,0,0,1,0]	(0,1,2,3,4,3,2,1,0,0,3)	2	19	1
(3,0,0)	[1,1,0,0,0,0,0,0,1]	(0,0,1,3,5,4,3,2,1,0,3)	2	22	1
(3,0,0)	[0,0,1,0,0,0,0,0,1]	(0,1,2,3,5,4,3,2,1,0,3)	0	24	0
(3,0,0)	[2,0,0,0,0,0,0,0,0]	(0,0,2,4,6,5,4,3,2,1,3)	0	30	0
(3,0,0)	[0,1,0,0,0,0,0,0,0]	(0,1,2,4,6,5,4,3,2,1,3)	-2	31	1
(0,1,0)	[0,0,0,0,0,0,0,0,1]	(0,0,0,0,0,0,0,0,0,1,0)	1	1	1
(1,1,0)	[0,0,1,0,0,0,0,0,0]	(0,0,0,0,1,1,1,1,1,1,1)	1	8	1
(2,1,0)	[1,0,0,0,0,1,0,0,0]	(0,0,1,2,3,2,1,1,1,1,2)	1	15	1
(2,1,0)	[0,0,0,0,0,0,1,0,0]	(0,1,2,3,4,3,2,1,1,1,2)	-1	21	1
(3,1,0)	[0,0,0,1,0,0,1,0,0]	(0,1,2,3,4,3,2,1,1,1,3)	1	22	1
(3,1,0)	[1,1,0,0,0,0,0,1,0]	(0,0,1,3,5,4,3,2,1,1,3)	1	24	1
(3,1,0)	[0,0,1,0,0,0,0,1,0]	(0,1,2,3,5,4,3,2,1,1,3)	-1	26	1
(3,1,0)	[2,0,0,0,0,0,0,0,1]	(0,0,2,4,6,5,4,3,2,1,3)	-1	31	1
(3,1,0)	[0,1,0,0,0,0,0,0,1]	(0,1,2,4,6,5,4,3,2,1,3)	-3	32	2
(3,1,0)	[1,0,0,0,0,0,0,0,0]	(0,1,3,5,7,6,5,4,3,2,1,3)	-5	40	2
(0,2,0)	[0,0,0,0,0,0,0,1,0]	(0,0,0,0,0,0,0,0,1,2,0)	2	3	1
(1,2,0)	[0,0,1,0,0,0,0,0,1]	(0,0,0,0,1,1,1,1,1,2,1)	2	9	1
(1,2,0)	[0,1,0,0,0,0,0,0,0]	(0,0,0,1,2,2,2,2,2,2,1)	0	16	1
(2,2,0)	[1,0,0,0,0,1,0,0,1]	(0,0,1,2,3,2,1,1,1,2,2)	2	16	1
(2,2,0)	[0,1,0,1,0,0,0,0,0]	(0,0,0,1,2,2,2,2,2,2,2)	2	17	1
(2,2,0)	[1,0,0,0,1,0,0,0,0]	(0,0,1,2,3,2,2,2,2,2,2)	0	20	1
(2,2,0)	[0,0,0,0,0,0,1,0,1]	(0,1,2,3,4,3,2,1,1,2,2)	0	22	1
(2,2,0)	[0,0,0,0,0,1,0,0,0]	(0,1,2,3,4,3,2,2,2,2,2)	-2	25	3
(3,2,0)	[1,0,1,0,0,1,0,0,0]	(0,0,1,2,4,3,2,2,2,2,3)	2	23	1
(3,2,0)	[0,0,0,1,0,0,1,0,1]	(0,1,2,3,4,3,2,1,1,2,3)	2	23	1
(3,2,0)	[1,1,0,0,0,0,0,1,1]	(0,0,1,3,5,4,3,2,1,2,3)	2	25	1
(3,2,0)	[0,0,0,1,0,1,0,0,0]	(0,1,2,3,4,3,2,2,2,2,3)	0	26	2
(3,2,0)	[1,1,0,0,0,0,1,0,0]	(0,0,1,3,5,4,3,2,2,2,3)	0	27	2
(3,2,0)	[0,0,1,0,0,0,0,1,1]	(0,1,2,3,5,4,3,2,1,2,3)	0	27	1
(3,2,0)	[0,0,1,0,0,0,1,0,0]	(0,1,2,3,5,4,3,2,2,2,3)	-2	29	4
(3,2,0)	[2,0,0,0,0,0,0,0,2]	(0,0,2,4,6,5,4,3,2,1,2,3)	0	32	1
(3,2,0)	[2,0,0,0,0,0,0,1,0]	(0,0,2,4,6,5,4,3,2,2,2,3)	-2	33	3
(3,2,0)	[0,1,0,0,0,0,0,0,2]	(0,1,2,4,6,5,4,3,2,1,2,3)	-2	33	2
(3,2,0)	[0,1,0,0,0,0,0,1,0]	(0,1,2,4,6,5,4,3,2,2,2,3)	-4	34	6
(3,2,0)	[1,0,0,0,0,0,0,0,1]	(0,1,3,5,7,6,5,4,3,2,2,3)	-6	41	7
(3,2,0)	[0,0,0,0,0,0,0,0,0]	(0,2,4,6,8,7,6,5,4,3,2,3)	-8	50	3
(1,3,0)	[0,1,0,0,0,0,0,0,1]	(0,0,0,1,2,2,2,2,2,3,1)	1	17	1

(n_c, n_d, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,3,0)	[1,0,0,0,0,0,0,0,0]	(0,0,1,2,3,3,3,3,3,3,3,1)	-1	25	1
(2,3,0)	[1,0,0,0,1,0,0,0,1]	(0,0,1,2,3,2,2,2,2,2,3,2)	1	21	1
(2,3,0)	[0,1,1,0,0,0,0,0,0]	(0,0,0,1,3,3,3,3,3,3,3,2)	1	24	1
(2,3,0)	[1,0,0,1,0,0,0,0,0]	(0,0,1,2,3,3,3,3,3,3,3,2)	-1	26	2
(2,3,0)	[0,0,0,0,0,1,0,0,1]	(0,1,2,3,4,3,2,2,2,2,3,2)	-1	26	2
(2,3,0)	[0,0,0,0,1,0,0,0,0]	(0,1,2,3,4,3,3,3,3,3,3,2)	-3	30	3
(0,0,1)	[1,0,0,0,0,0,0,0,0]	(1,0,0,0,0,0,0,0,0,0,0,0)	2	1	1
(1,0,1)	[0,0,0,0,1,0,0,0,0]	(1,1,1,1,1,0,0,0,0,0,0,1)	2	6	1
(2,0,1)	[0,1,0,0,0,0,1,0,0]	(1,1,1,2,3,2,1,0,0,0,0,2)	2	13	1
(2,0,1)	[1,0,0,0,0,0,0,1,0]	(1,1,2,3,4,3,2,1,0,0,0,2)	0	19	1
(2,0,1)	[0,0,0,0,0,0,0,0,1]	(1,2,3,4,5,4,3,2,1,0,0,2)	-2	27	1
(3,0,1)	[1,0,0,1,0,0,0,1,0]	(1,1,2,3,4,3,2,1,0,0,0,3)	2	20	1
(3,0,1)	[0,0,0,0,0,1,1,0,0]	(1,2,3,4,5,3,1,0,0,0,0,3)	2	22	1
(3,0,1)	[0,2,0,0,0,0,0,0,1]	(1,1,1,3,5,4,3,2,1,0,0,3)	2	24	1
(3,0,1)	[0,0,0,0,1,0,0,1,0]	(1,2,3,4,5,3,2,1,0,0,0,3)	0	24	1
(3,0,1)	[1,0,1,0,0,0,0,0,1]	(1,1,2,3,5,4,3,2,1,0,0,3)	0	25	1
(3,0,1)	[0,0,0,1,0,0,0,0,1]	(1,2,3,4,5,4,3,2,1,0,0,3)	-2	28	2
(3,0,1)	[1,1,0,0,0,0,0,0,0]	(1,1,2,4,6,5,4,3,2,1,0,3)	-2	32	2
(3,0,1)	[0,0,1,0,0,0,0,0,0]	(1,2,3,4,6,5,4,3,2,1,0,3)	-4	34	2
(0,1,1)	[0,0,0,0,0,0,0,0,0]	(1,1,1,1,1,1,1,1,1,1,1,0)	1	11	1
(1,1,1)	[0,0,0,1,0,0,0,0,0]	(1,1,1,1,1,1,1,1,1,1,1,1)	1	12	1
(2,1,1)	[0,1,0,0,0,1,0,0,0]	(1,1,1,2,3,2,1,1,1,1,1,2)	1	17	1
(2,1,1)	[1,0,0,0,0,0,1,0,0]	(1,1,2,3,4,3,2,1,1,1,1,2)	-1	22	2
(2,1,1)	[0,0,0,0,0,0,0,1,0]	(1,2,3,4,5,4,3,2,1,1,1,2)	-3	29	2
(3,1,1)	[1,0,0,1,0,0,1,0,0]	(1,1,2,3,4,3,2,1,1,1,1,3)	1	23	1
(3,1,1)	[0,2,0,0,0,0,0,1,0]	(1,1,1,3,5,4,3,2,1,1,1,3)	1	26	1
(3,1,1)	[0,0,0,0,0,2,0,0,0]	(1,2,3,4,5,3,1,1,1,1,1,3)	1	26	1
(3,1,1)	[1,0,1,0,0,0,0,1,0]	(1,1,2,3,5,4,3,2,1,1,1,3)	-1	27	2
(3,1,1)	[0,0,0,0,1,0,1,0,0]	(1,2,3,4,5,3,2,1,1,1,1,3)	-1	27	2
(3,1,1)	[0,0,0,1,0,0,0,1,0]	(1,2,3,4,5,4,3,2,1,1,1,3)	-3	30	4
(3,1,1)	[1,1,0,0,0,0,0,0,1]	(1,1,2,4,6,5,4,3,2,1,1,3)	-3	33	4
(3,1,1)	[0,0,1,0,0,0,0,0,1]	(1,2,3,4,6,5,4,3,2,1,1,3)	-5	35	5
(3,1,1)	[2,0,0,0,0,0,0,0,0]	(1,1,3,5,7,6,5,4,3,2,1,3)	-5	41	3
(3,1,1)	[0,1,0,0,0,0,0,0,0]	(1,2,3,5,7,6,5,4,3,2,1,3)	-7	42	7
(0,2,1)	[0,0,0,0,0,0,0,0,1]	(1,1,1,1,1,1,1,1,1,1,2,0)	2	12	1
(1,2,1)	[0,0,0,1,0,0,0,0,1]	(1,1,1,1,1,1,1,1,1,1,2,1)	2	13	1
(1,2,1)	[0,0,1,0,0,0,0,0,0]	(1,1,1,1,2,2,2,2,2,2,2,1)	0	19	2
(2,2,1)	[0,1,0,0,0,1,0,0,1]	(1,1,1,2,3,2,1,1,1,1,2,2)	2	18	1
(2,2,1)	[0,0,1,1,0,0,0,0,0]	(1,1,1,1,2,2,2,2,2,2,2,2)	2	20	1
(2,2,1)	[0,1,0,0,1,0,0,0,0]	(1,1,1,2,3,2,2,2,2,2,2,2)	0	22	2
(2,2,1)	[1,0,0,0,0,0,1,0,1]	(1,1,2,3,4,3,2,1,1,1,2,2)	0	23	2

(n_c, n_d, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(2,2,1)	[1,0,0,0,0,1,0,0,0]	(1,1,2,3,4,3,2,2,2,2,2,2)	-2	26	5
(2,2,1)	[0,0,0,0,0,0,0,1,1]	(1,2,3,4,5,4,3,2,1,1,2,2)	-2	30	2
(2,2,1)	[0,0,0,0,0,0,1,0,0]	(1,2,3,4,5,4,3,2,2,2,2,2)	-4	32	6
(1,3,1)	[0,0,1,0,0,0,0,0,1]	(1,1,1,1,2,2,2,2,2,2,3,1)	1	20	1
(1,3,1)	[0,1,0,0,0,0,0,0,0]	(1,1,1,2,3,3,3,3,3,3,3,1)	-1	27	2
(2,3,1)	[0,1,0,0,1,0,0,0,1]	(1,1,1,2,3,2,2,2,2,2,3,2)	1	23	1
(2,3,1)	[1,0,0,0,0,1,0,0,1]	(1,1,2,3,4,3,2,2,2,2,3,2)	-1	27	3
(2,3,1)	[0,0,2,0,0,0,0,0,0]	(1,1,1,1,3,3,3,3,3,3,3,2)	1	27	1
(2,3,1)	[0,1,0,1,0,0,0,0,0]	(1,1,1,2,3,3,3,3,3,3,3,2)	-1	28	3
(2,3,1)	[1,0,0,0,1,0,0,0,0]	(1,1,2,3,4,3,3,3,3,3,3,2)	-3	31	6
(2,3,1)	[0,0,0,0,0,0,0,1,2]	(1,2,3,4,5,4,3,2,1,1,3,2)	1	31	1
(2,3,1)	[0,0,0,0,0,0,1,0,1]	(1,2,3,4,5,4,3,2,2,2,3,2)	-3	33	2
(2,3,1)	[0,0,0,0,0,1,0,0,0]	(1,2,3,4,5,4,3,3,3,3,3,2)	-5	36	12

B.8 A_1^{+++1}

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,0)	[0,0,2]	(0,0,0,0,1)	2	1	1
(1,0)	[0,1,0]	(0,0,0,1,1)	0	2	0
(2,0)	[0,1,2]	(0,0,0,1,2)	2	3	1
(2,0)	[0,2,0]	(0,0,0,2,2)	0	4	0
(2,0)	[1,0,1]	(0,0,1,2,2)	-2	5	0
(2,0)	[0,0,0]	(0,1,2,3,2)	-4	8	0
(3,0)	[0,2,2]	(0,0,0,2,3)	2	5	1
(3,0)	[1,0,3]	(0,0,1,2,3)	0	6	0
(3,0)	[0,3,0]	(0,0,0,3,3)	0	6	0
(3,0)	[1,1,1]	(0,0,1,3,3)	-4	7	1
(3,0)	[2,0,0]	(0,0,2,4,3)	-6	9	0
(3,0)	[0,0,2]	(0,1,2,3,3)	-6	9	0
(3,0)	[0,1,0]	(0,1,2,4,3)	-8	10	0
(4,0)	[0,3,2]	(0,0,0,3,4)	2	7	1
(4,0)	[1,1,3]	(0,0,1,3,4)	-2	8	1
(4,0)	[0,4,0]	(0,0,0,4,4)	0	8	0
(4,0)	[1,2,1]	(0,0,1,4,4)	-6	9	2
(4,0)	[2,0,2]	(0,0,2,4,4)	-8	10	1
(4,0)	[0,0,4]	(0,1,2,3,4)	-4	10	0
(4,0)	[2,1,0]	(0,0,2,5,4)	-10	11	1
(4,0)	[0,1,2]	(0,1,2,4,4)	-10	11	1
(4,0)	[0,2,0]	(0,1,2,5,4)	-12	12	0
(4,0)	[1,0,1]	(0,1,3,5,4)	-14	13	1
(4,0)	[0,0,0]	(0,2,4,6,4)	-16	16	0
(5,0)	[0,4,2]	(0,0,0,4,5)	2	9	1

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(5,0)	[1,2,3]	(0,0,1,4,5)	-4	10	2
(5,0)	[0,5,0]	(0,0,0,5,5)	0	10	0
(5,0)	[2,0,4]	(0,0,2,4,5)	-6	11	2
(5,0)	[1,3,1]	(0,0,1,5,5)	-8	11	3
(5,0)	[2,1,2]	(0,0,2,5,5)	-12	12	4
(5,0)	[0,1,4]	(0,1,2,4,5)	-8	12	1
(5,0)	[2,2,0]	(0,0,2,6,5)	-14	13	3
(5,0)	[0,2,2]	(0,1,2,5,5)	-14	13	3
(5,0)	[3,0,1]	(0,0,3,6,5)	-16	14	2
(5,0)	[1,0,3]	(0,1,3,5,5)	-16	14	3
(5,0)	[0,3,0]	(0,1,2,6,5)	-16	14	2
(5,0)	[1,1,1]	(0,1,3,6,5)	-20	15	5
(5,0)	[2,0,0]	(0,1,4,7,5)	-22	17	1
(5,0)	[0,0,2]	(0,2,4,6,5)	-22	17	2
(5,0)	[0,1,0]	(0,2,4,7,5)	-24	18	1
(0,1)	[1,0,0]	(1,0,0,0,0)	2	1	1
(1,1)	[0,0,1]	(1,1,1,1,1)	0	5	1
(2,1)	[0,0,3]	(1,1,1,1,2)	2	6	1
(2,1)	[0,1,1]	(1,1,1,2,2)	-2	7	1
(2,1)	[1,0,0]	(1,1,2,3,2)	-4	9	0
(3,1)	[0,1,3]	(1,1,1,2,3)	0	8	1
(3,1)	[0,2,1]	(1,1,1,3,3)	-4	9	2
(3,1)	[1,0,2]	(1,1,2,3,3)	-6	10	1
(3,1)	[1,1,0]	(1,1,2,4,3)	-8	11	1
(3,1)	[0,0,1]	(1,2,3,4,3)	-10	13	0
(4,1)	[0,2,3]	(1,1,1,3,4)	-2	10	2
(4,1)	[1,0,4]	(1,1,2,3,4)	-4	11	1
(4,1)	[0,3,1]	(1,1,1,4,4)	-6	11	3
(4,1)	[1,1,2]	(1,1,2,4,4)	-10	12	5
(4,1)	[1,2,0]	(1,1,2,5,4)	-12	13	3
(4,1)	[2,0,1]	(1,1,3,5,4)	-14	14	3
(4,1)	[0,0,3]	(1,2,3,4,4)	-12	14	1
(4,1)	[0,1,1]	(1,2,3,5,4)	-16	15	2
(4,1)	[1,0,0]	(1,2,4,6,4)	-18	17	1

C. The $\bar{l}_1$ representation of E_8^{++}

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(1,0)	[0,0,0,0,0,0,1,0,0]	(0,0,0,0,0,0,0,0,0,0,1)	2	1	1
(2,0)	[0,0,0,1,0,0,0,0,0]	(0,0,0,0,0,1,2,3,2,1,2)	2	11	1
(3,0)	[0,1,0,0,0,0,0,0,1]	(0,0,0,1,2,3,4,5,3,1,3)	2	22	1

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(3,0)	[1,0,0,0,0,0,0,0,0]	(0,0,1,2,3,4,5,6,4,2,3)	0	30	0
(4,0)	[1,0,0,0,0,0,1,0,0]	(0,0,1,2,3,4,5,6,4,2,4)	2	31	1
(4,0)	[0,0,0,0,0,0,0,0,2]	(0,1,2,3,4,5,6,7,4,1,4)	2	37	1
(4,0)	[0,0,0,0,0,0,0,1,0]	(0,1,2,3,4,5,6,7,4,2,4)	0	38	0
(5,0)	[1,0,0,1,0,0,0,0,0]	(0,0,1,2,3,5,7,9,6,3,5)	2	41	1
(5,0)	[0,0,0,0,0,1,0,0,1]	(0,1,2,3,4,5,6,8,5,2,5)	2	41	1
(5,0)	[0,0,0,0,1,0,0,0,0]	(0,1,2,3,4,5,7,9,6,3,5)	0	45	0
(6,0)	[0,0,0,1,0,0,0,1,0]	(0,1,2,3,4,6,8,10,6,3,6)	2	49	1
(6,0)	[1,1,0,0,0,0,0,0,1]	(0,0,1,3,5,7,9,11,7,3,6)	2	52	1
(6,0)	[0,0,1,0,0,0,0,0,1]	(0,1,2,3,5,7,9,11,7,3,6)	0	54	1
(6,0)	[2,0,0,0,0,0,0,0,0]	(0,0,2,4,6,8,10,12,8,4,6)	0	60	0
(6,0)	[0,1,0,0,0,0,0,0,0]	(0,1,2,4,6,8,10,12,8,4,6)	-2	61	1
(7,0)	[0,0,1,0,0,1,0,0,0]	(0,1,2,3,5,7,9,12,8,4,7)	2	58	1
(7,0)	[0,1,0,0,0,0,0,1,1]	(0,1,2,4,6,8,10,12,7,3,7)	2	60	1
(7,0)	[2,0,0,0,0,0,1,0,0]	(0,0,2,4,6,8,10,12,8,4,7)	2	61	1
(7,0)	[0,1,0,0,0,0,1,0,0]	(0,1,2,4,6,8,10,12,8,4,7)	0	62	1
(7,0)	[1,0,0,0,0,0,0,0,2]	(0,1,3,5,7,9,11,13,8,3,7)	0	67	1
(7,0)	[1,0,0,0,0,0,0,1,0]	(0,1,3,5,7,9,11,13,8,4,7)	-2	68	2
(7,0)	[0,0,0,0,0,0,0,0,1]	(0,2,4,6,8,10,12,14,9,4,7)	-4	76	1
(8,0)	[0,1,0,0,1,0,0,0,1]	(0,1,2,4,6,8,11,14,9,4,8)	2	67	1
(8,0)	[1,0,0,0,0,0,1,1,0]	(0,1,3,5,7,9,11,13,8,4,8)	2	69	1
(8,0)	[2,0,0,1,0,0,0,0,0]	(0,0,2,4,6,9,12,15,10,5,8)	2	71	1
(8,0)	[1,0,0,0,0,1,0,0,1]	(0,1,3,5,7,9,11,14,9,4,8)	0	71	2
(8,0)	[0,0,2,0,0,0,0,0,0]	(0,1,2,3,6,9,12,15,10,5,8)	2	71	1
(8,0)	[0,1,0,1,0,0,0,0,0]	(0,1,2,4,6,9,12,15,10,5,8)	0	72	1
(8,0)	[1,0,0,0,1,0,0,0,0]	(0,1,3,5,7,9,12,15,10,5,8)	-2	75	2
(8,0)	[0,0,0,0,0,0,0,1,2]	(0,2,4,6,8,10,12,14,8,3,8)	2	75	1
(8,0)	[0,0,0,0,0,0,0,2,0]	(0,2,4,6,8,10,12,14,8,4,8)	0	76	0
(8,0)	[0,0,0,0,0,0,1,0,1]	(0,2,4,6,8,10,12,14,9,4,8)	-2	77	2
(8,0)	[0,0,0,0,0,1,0,0,0]	(0,2,4,6,8,10,12,15,10,5,8)	-4	80	2
(9,0)	[1,0,0,0,1,0,1,0,0]	(0,1,3,5,7,9,12,15,10,5,9)	2	76	1
(9,0)	[0,1,1,0,0,0,0,1,0]	(0,1,2,4,7,10,13,16,10,5,9)	2	77	1
(9,0)	[1,0,0,1,0,0,0,0,2]	(0,1,3,5,7,10,13,16,10,4,9)	2	78	1
(9,0)	[1,0,0,1,0,0,0,1,0]	(0,1,3,5,7,10,13,16,10,5,9)	0	79	2
(9,0)	[0,0,0,0,0,1,0,1,1]	(0,2,4,6,8,10,12,15,9,4,9)	2	79	1
(9,0)	[0,0,0,0,0,1,1,0,0]	(0,2,4,6,8,10,12,15,10,5,9)	0	81	1
(9,0)	[2,1,0,0,0,0,0,0,1]	(0,0,2,5,8,11,14,17,11,5,9)	2	82	1
(9,0)	[0,0,0,0,1,0,0,0,2]	(0,2,4,6,8,10,13,16,10,4,9)	0	82	1
(9,0)	[0,2,0,0,0,0,0,0,1]	(0,1,2,5,8,11,14,17,11,5,9)	0	83	1
(9,0)	[0,0,0,0,1,0,0,1,0]	(0,2,4,6,8,10,13,16,10,5,9)	-2	83	3
(9,0)	[1,0,1,0,0,0,0,0,1]	(0,1,3,5,8,11,14,17,11,5,9)	-2	84	4

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(9,0)	[0,0,0,1,0,0,0,0,1]	(0,2,4,6,8,11,14,17,11,5,9)	−4	87	4
(9,0)	[3,0,0,0,0,0,0,0,0]	(0,0,3,6,9,12,15,18,12,6,9)	0	90	0
(9,0)	[1,1,0,0,0,0,0,0,0]	(0,1,3,6,9,12,15,18,12,6,9)	−4	91	3
(9,0)	[0,0,1,0,0,0,0,0,0]	(0,2,4,6,9,12,15,18,12,6,9)	−6	93	4
(10,0)	[1,0,1,0,0,0,1,0,1]	(0,1,3,5,8,11,14,17,11,5,10)	2	85	1
(10,0)	[1,0,0,1,1,0,0,0,0]	(0,1,3,5,7,10,14,18,12,6,10)	2	86	1
(10,0)	[0,0,0,0,1,1,0,0,1]	(0,2,4,6,8,10,13,17,11,5,10)	2	86	1
(10,0)	[0,2,0,0,0,1,0,0,0]	(0,1,2,5,8,11,14,18,12,6,10)	2	87	1
(10,0)	[0,0,0,1,0,0,0,2,0]	(0,2,4,6,8,11,14,17,10,5,10)	2	87	1
(10,0)	[1,0,1,0,0,1,0,0,0]	(0,1,3,5,8,11,14,18,12,6,10)	0	88	2
(10,0)	[0,0,0,1,0,0,1,0,1]	(0,2,4,6,8,11,14,17,11,5,10)	0	88	2
(10,0)	[1,1,0,0,0,0,0,1,1]	(0,1,3,6,9,12,15,18,11,5,10)	0	90	3
(10,0)	[0,0,0,0,2,0,0,0,0]	(0,2,4,6,8,10,14,18,12,6,10)	0	90	0
(10,0)	[3,0,0,0,0,0,1,0,0]	(0,0,3,6,9,12,15,18,12,6,10)	2	91	1
(10,0)	[0,0,1,0,0,0,0,0,3]	(0,2,4,6,9,12,15,18,11,4,10)	2	91	1
(10,0)	[0,0,0,1,0,1,0,0,0]	(0,2,4,6,8,11,14,18,12,6,10)	−2	91	3
(10,0)	[1,1,0,0,0,0,1,0,0]	(0,1,3,6,9,12,15,18,12,6,10)	−2	92	4
(10,0)	[0,0,1,0,0,0,0,1,1]	(0,2,4,6,9,12,15,18,11,5,10)	−2	92	4
(10,0)	[0,0,1,0,0,0,1,0,0]	(0,2,4,6,9,12,15,18,12,6,10)	−4	94	6
(10,0)	[2,0,0,0,0,0,0,0,2]	(0,1,4,7,10,13,16,19,12,5,10)	−2	97	3
(10,0)	[2,0,0,0,0,0,0,1,0]	(0,1,4,7,10,13,16,19,12,6,10)	−4	98	4
(10,0)	[0,1,0,0,0,0,0,0,2]	(0,2,4,7,10,13,16,19,12,5,10)	−4	98	4
(10,0)	[0,1,0,0,0,0,0,1,0]	(0,2,4,7,10,13,16,19,12,6,10)	−6	99	10
(10,0)	[1,0,0,0,0,0,0,0,1]	(0,2,5,8,11,14,17,20,13,6,10)	−8	106	8
(10,0)	[0,0,0,0,0,0,0,0,0]	(0,3,6,9,12,15,18,21,14,7,10)	−10	115	3
(11,0)	[1,1,0,0,0,1,0,1,0]	(0,1,3,6,9,12,15,19,12,6,11)	2	94	1
(11,0)	[0,0,0,1,1,0,0,1,0]	(0,2,4,6,8,11,15,19,12,6,11)	2	94	1
(11,0)	[1,0,1,1,0,0,0,0,1]	(0,1,3,5,8,12,16,20,13,6,11)	2	95	1
(11,0)	[0,0,1,0,0,1,0,0,2]	(0,2,4,6,9,12,15,19,12,5,11)	2	95	1
(11,0)	[0,0,1,0,0,0,2,0,0]	(0,2,4,6,9,12,15,18,12,6,11)	2	95	1
(11,0)	[0,0,1,0,0,1,0,1,0]	(0,2,4,6,9,12,15,19,12,6,11)	0	96	2
(11,0)	[1,1,0,0,1,0,0,0,1]	(0,1,3,6,9,12,16,20,13,6,11)	0	97	3
(11,0)	[2,0,0,0,0,0,1,0,2]	(0,1,4,7,10,13,16,19,12,5,11)	2	98	1
(11,0)	[0,1,0,0,0,0,0,2,1]	(0,2,4,7,10,13,16,19,11,5,11)	2	98	1
(11,0)	[0,0,0,2,0,0,0,0,1]	(0,2,4,6,8,12,16,20,13,6,11)	0	98	1
(11,0)	[2,0,0,0,0,0,1,1,0]	(0,1,4,7,10,13,16,19,12,6,11)	0	99	2
(11,0)	[0,1,0,0,0,0,1,0,2]	(0,2,4,7,10,13,16,19,12,5,11)	0	99	2
(11,0)	[0,0,1,0,1,0,0,0,1]	(0,2,4,6,9,12,16,20,13,6,11)	−2	99	5
(11,0)	[0,2,1,0,0,0,0,0,0]	(0,1,2,5,9,13,17,21,14,7,11)	2	100	1
(11,0)	[0,1,0,0,0,0,1,1,0]	(0,2,4,7,10,13,16,19,12,6,11)	−2	100	5
(11,0)	[3,0,0,1,0,0,0,0,0]	(0,0,3,6,9,13,17,21,14,7,11)	2	101	1

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(11,0)	[2,0,0,0,1,0,0,1]	(0,1,4,7,10,13,16,20,13,6,11)	-2	101	5
(11,0)	[1,0,2,0,0,0,0,0,0]	(0,1,3,5,9,13,17,21,14,7,11)	0	101	1
(11,0)	[1,1,0,1,0,0,0,0,0]	(0,1,3,6,9,13,17,21,14,7,11)	-2	102	4
(11,0)	[0,1,0,0,0,1,0,0,1]	(0,2,4,7,10,13,16,20,13,6,11)	-4	102	10
(11,0)	[0,0,1,1,0,0,0,0,0]	(0,2,4,6,9,13,17,21,14,7,11)	-4	104	5
(11,0)	[2,0,0,0,1,0,0,0,0]	(0,1,4,7,10,13,17,21,14,7,11)	-4	105	5
(11,0)	[1,0,0,0,0,0,0,1,2]	(0,2,5,8,11,14,17,20,12,5,11)	-2	105	4
(11,0)	[1,0,0,0,0,0,0,2,0]	(0,2,5,8,11,14,17,20,12,6,11)	-4	106	5
(11,0)	[0,1,0,0,1,0,0,0,0]	(0,2,4,7,10,13,17,21,14,7,11)	-6	106	11
(11,0)	[1,0,0,0,0,0,1,0,1]	(0,2,5,8,11,14,17,20,13,6,11)	-6	107	15
(11,0)	[1,0,0,0,0,1,0,0,0]	(0,2,5,8,11,14,17,21,14,7,11)	-8	110	14
(11,0)	[0,0,0,0,0,0,0,0,3]	(0,3,6,9,12,15,18,21,13,5,11)	-4	113	2
(11,0)	[0,0,0,0,0,0,0,1,1]	(0,3,6,9,12,15,18,21,13,6,11)	-8	114	9
(11,0)	[0,0,0,0,0,0,1,0,0]	(0,3,6,9,12,15,18,21,14,7,11)	-10	116	12
(12,0)	[1,1,0,1,0,0,1,0,0]	(0,1,3,6,9,13,17,21,14,7,12)	2	103	1
(12,0)	[0,1,0,0,0,1,1,0,1]	(0,2,4,7,10,13,16,20,13,6,12)	2	103	1
(12,0)	[0,0,1,1,0,0,0,1,1]	(0,2,4,6,9,13,17,21,13,6,12)	2	103	1
(12,0)	[0,0,1,0,1,1,0,0,0]	(0,2,4,6,9,12,16,21,14,7,12)	2	103	1
(12,0)	[2,0,0,0,1,0,0,1,1]	(0,1,4,7,10,13,17,21,13,6,12)	2	104	1
(12,0)	[2,0,0,0,0,2,0,0,0]	(0,1,4,7,10,13,16,21,14,7,12)	2	105	1
(12,0)	[0,1,0,0,1,0,0,1,1]	(0,2,4,7,10,13,17,21,13,6,12)	0	105	3
(12,0)	[0,0,1,1,0,0,1,0,0]	(0,2,4,6,9,13,17,21,14,7,12)	0	105	2
(12,0)	[2,0,0,0,1,0,1,0,0]	(0,1,4,7,10,13,17,21,14,7,12)	0	106	2
(12,0)	[1,1,1,0,0,0,0,0,2]	(0,1,3,6,10,14,18,22,14,6,12)	2	106	1
(12,0)	[0,1,0,0,0,2,0,0,0]	(0,2,4,7,10,13,16,21,14,7,12)	0	106	1
(12,0)	[1,1,1,0,0,0,0,1,0]	(0,1,3,6,10,14,18,22,14,7,12)	0	107	3
(12,0)	[1,0,0,0,0,0,1,2,0]	(0,2,5,8,11,14,17,20,12,6,12)	2	107	1
(12,0)	[0,1,0,0,1,0,1,0,0]	(0,2,4,7,10,13,17,21,14,7,12)	-2	107	6
(12,0)	[2,0,0,1,0,0,0,0,2]	(0,1,4,7,10,14,18,22,14,6,12)	0	108	2
(12,0)	[1,0,0,0,0,1,0,0,3]	(0,2,5,8,11,14,17,21,13,5,12)	2	108	1
(12,0)	[1,0,0,0,0,0,2,0,1]	(0,2,5,8,11,14,17,20,13,6,12)	0	108	2
(12,0)	[0,0,2,0,0,0,0,0,2]	(0,2,4,6,10,14,18,22,14,6,12)	0	108	1
(12,0)	[2,0,0,1,0,0,0,1,0]	(0,1,4,7,10,14,18,22,14,7,12)	-2	109	6
(12,0)	[1,0,0,0,0,1,0,1,1]	(0,2,5,8,11,14,17,21,13,6,12)	-2	109	7
(12,0)	[0,1,0,1,0,0,0,0,2]	(0,2,4,7,10,14,18,22,14,6,12)	-2	109	6
(12,0)	[0,0,2,0,0,0,0,1,0]	(0,2,4,6,10,14,18,22,14,7,12)	-2	109	4
(12,0)	[0,1,0,1,0,0,0,1,0]	(0,2,4,7,10,14,18,22,14,7,12)	-4	110	11
(12,0)	[1,0,0,0,0,1,1,0,0]	(0,2,5,8,11,14,17,21,14,7,12)	-4	111	9
(12,0)	[3,1,0,0,0,0,0,0,1]	(0,0,3,7,11,15,19,23,15,7,12)	2	112	1
(12,0)	[1,0,0,0,1,0,0,0,2]	(0,2,5,8,11,14,18,22,14,6,12)	-4	112	10
(12,0)	[1,2,0,0,0,0,0,0,1]	(0,1,3,7,11,15,19,23,15,7,12)	-2	113	4

(n_c,n_0)	p_i	β	β^2	$ht(\beta)$	μ
(12,0)	[1,0,0,0,1,0,0,1,0]	(0,2,5,8,11,14,18,22,14,7,12)	-6	113	21
(12,0)	[0,0,0,0,0,0,0,2,2]	(0,3,6,9,12,15,18,21,12,5,12)	2	113	1
(12,0)	[2,0,1,0,0,0,0,0,1]	(0,1,4,7,11,15,19,23,15,7,12)	-4	114	9
(12,0)	[0,0,0,0,0,0,1,0,3]	(0,3,6,9,12,15,18,21,13,5,12)	0	114	1
(12,0)	[0,0,0,0,0,0,0,3,0]	(0,3,6,9,12,15,18,21,12,6,12)	0	114	0
(12,0)	[0,1,1,0,0,0,0,0,1]	(0,2,4,7,11,15,19,23,15,7,12)	-6	115	17
(12,0)	[0,0,0,0,0,0,1,1,1]	(0,3,6,9,12,15,18,21,13,6,12)	-4	115	7
(12,0)	[1,0,0,1,0,0,0,0,1]	(0,2,5,8,11,15,19,23,15,7,12)	-8	117	30
(12,0)	[0,0,0,0,0,1,0,0,2]	(0,3,6,9,12,15,18,22,14,6,12)	-6	117	11
(12,0)	[0,0,0,0,0,0,2,0,0]	(0,3,6,9,12,15,18,21,14,7,12)	-6	117	7
(12,0)	[0,0,0,0,0,1,0,1,0]	(0,3,6,9,12,15,18,22,14,7,12)	-8	118	17
(12,0)	[4,0,0,0,0,0,0,0,0]	(0,0,4,8,12,16,20,24,16,8,12)	0	120	0
(12,0)	[2,1,0,0,0,0,0,0,0]	(0,1,4,8,12,16,20,24,16,8,12)	-6	121	7
(12,0)	[0,0,0,0,1,0,0,0,1]	(0,3,6,9,12,15,19,23,15,7,12)	-10	121	27
(12,0)	[0,2,0,0,0,0,0,0,0]	(0,2,4,8,12,16,20,24,16,8,12)	-8	122	8
(12,0)	[1,0,1,0,0,0,0,0,0]	(0,2,5,8,12,16,20,24,16,8,12)	-10	123	22
(12,0)	[0,0,0,1,0,0,0,0,0]	(0,3,6,9,12,16,20,24,16,8,12)	-12	126	18
(0,1)	[1,0,0,0,0,0,0,0,0]	(1,0,0,0,0,0,0,0,0,0,0)	2	1	1
(1,1)	[0,0,0,0,0,0,0,1,0]	(1,1,1,1,1,1,1,1,0,0,1)	2	9	1
(2,1)	[0,0,0,0,1,0,0,0,0]	(1,1,1,1,1,1,2,3,2,1,2)	2	16	1
(3,1)	[0,0,1,0,0,0,0,0,1]	(1,1,1,1,2,3,4,5,3,1,3)	2	25	1
(3,1)	[0,1,0,0,0,0,0,0,0]	(1,1,1,2,3,4,5,6,4,2,3)	0	32	1
(4,1)	[0,1,0,0,0,0,1,0,0]	(1,1,1,2,3,4,5,6,4,2,4)	2	33	1
(4,1)	[1,0,0,0,0,0,0,0,2]	(1,1,2,3,4,5,6,7,4,1,4)	2	38	1
(4,1)	[1,0,0,0,0,0,0,1,0]	(1,1,2,3,4,5,6,7,4,2,4)	0	39	1
(4,1)	[0,0,0,0,0,0,0,0,1]	(1,2,3,4,5,6,7,8,5,2,4)	-2	47	2
(5,1)	[1,0,0,0,0,1,0,0,1]	(1,1,2,3,4,5,6,8,5,2,5)	2	42	1
(5,1)	[0,1,0,1,0,0,0,0,0]	(1,1,1,2,3,5,7,9,6,3,5)	2	43	1
(5,1)	[1,0,0,0,1,0,0,0,0]	(1,1,2,3,4,5,7,9,6,3,5)	0	46	1
(5,1)	[0,0,0,0,0,0,1,0,1]	(1,2,3,4,5,6,7,8,5,2,5)	0	48	2
(5,1)	[0,0,0,0,0,1,0,0,0]	(1,2,3,4,5,6,7,9,6,3,5)	-2	51	2
(6,1)	[1,0,0,1,0,0,0,1,0]	(1,1,2,3,4,6,8,10,6,3,6)	2	50	1
(6,1)	[0,0,0,0,0,1,1,0,0]	(1,2,3,4,5,6,7,9,6,3,6)	2	52	1
(6,1)	[0,0,0,0,1,0,0,0,2]	(1,2,3,4,5,6,8,10,6,2,6)	2	53	1
(6,1)	[0,2,0,0,0,0,0,0,1]	(1,1,1,3,5,7,9,11,7,3,6)	2	54	1
(6,1)	[0,0,0,0,1,0,0,1,0]	(1,2,3,4,5,6,8,10,6,3,6)	0	54	1
(6,1)	[1,0,1,0,0,0,0,0,1]	(1,1,2,3,5,7,9,11,7,3,6)	0	55	2
(6,1)	[0,0,0,1,0,0,0,0,1]	(1,2,3,4,5,7,9,11,7,3,6)	-2	58	4
(6,1)	[1,1,0,0,0,0,0,0,0]	(1,1,2,4,6,8,10,12,8,4,6)	-2	62	2
(6,1)	[0,0,1,0,0,0,0,0,0]	(1,2,3,4,6,8,10,12,8,4,6)	-4	64	3
(7,1)	[1,0,1,0,0,1,0,0,0]	(1,1,2,3,5,7,9,12,8,4,7)	2	59	1

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(7,1)	[0,0,0,1,0,0,1,0,1]	(1,2,3,4,5,7,9,11,7,3,7)	2	59	1
(7,1)	[1,1,0,0,0,0,0,1,1]	(1,1,2,4,6,8,10,12,7,3,7)	2	61	1
(7,1)	[0,0,0,1,0,1,0,0,0]	(1,2,3,4,5,7,9,12,8,4,7)	0	62	2
(7,1)	[1,1,0,0,0,0,1,0,0]	(1,1,2,4,6,8,10,12,8,4,7)	0	63	2
(7,1)	[0,0,1,0,0,0,0,1,1]	(1,2,3,4,6,8,10,12,7,3,7)	0	63	2
(7,1)	[0,0,1,0,0,0,1,0,0]	(1,2,3,4,6,8,10,12,8,4,7)	-2	65	4
(7,1)	[2,0,0,0,0,0,0,0,2]	(1,1,3,5,7,9,11,13,8,3,7)	0	68	1
(7,1)	[2,0,0,0,0,0,0,1,0]	(1,1,3,5,7,9,11,13,8,4,7)	-2	69	3
(7,1)	[0,1,0,0,0,0,0,0,2]	(1,2,3,5,7,9,11,13,8,3,7)	-2	69	5
(7,1)	[0,1,0,0,0,0,0,1,0]	(1,2,3,5,7,9,11,13,8,4,7)	-4	70	7
(7,1)	[1,0,0,0,0,0,0,0,1]	(1,2,4,6,8,10,12,14,9,4,7)	-6	77	8
(7,1)	[0,0,0,0,0,0,0,0,0]	(1,3,5,7,9,11,13,15,10,5,7)	-8	86	2
(8,1)	[0,0,1,0,0,1,0,1,0]	(1,2,3,4,6,8,10,13,8,4,8)	2	67	1
(8,1)	[1,1,0,0,1,0,0,0,1]	(1,1,2,4,6,8,11,14,9,4,8)	2	68	1
(8,1)	[0,0,0,2,0,0,0,0,1]	(1,2,3,4,5,8,11,14,9,4,8)	2	69	1
(8,1)	[2,0,0,0,0,0,1,1,0]	(1,1,3,5,7,9,11,13,8,4,8)	2	70	1
(8,1)	[0,1,0,0,0,0,1,0,2]	(1,2,3,5,7,9,11,13,8,3,8)	2	70	1
(8,1)	[0,0,1,0,1,0,0,0,1]	(1,2,3,4,6,8,11,14,9,4,8)	0	70	2
(8,1)	[0,1,0,0,0,0,1,1,0]	(1,2,3,5,7,9,11,13,8,4,8)	0	71	2
(8,1)	[2,0,0,0,0,1,0,0,1]	(1,1,3,5,7,9,11,14,9,4,8)	0	72	2
(8,1)	[1,0,2,0,0,0,0,0,0]	(1,1,2,3,6,9,12,15,10,5,8)	2	72	1
(8,1)	[1,1,0,1,0,0,0,0,0]	(1,1,2,4,6,9,12,15,10,5,8)	0	73	2
(8,1)	[0,1,0,0,0,1,0,0,1]	(1,2,3,5,7,9,11,14,9,4,8)	-2	73	7
(8,1)	[0,0,1,1,0,0,0,0,0]	(1,2,3,4,6,9,12,15,10,5,8)	-2	75	4
(8,1)	[2,0,0,0,1,0,0,0,0]	(1,1,3,5,7,9,12,15,10,5,8)	-2	76	3
(8,1)	[1,0,0,0,0,0,0,1,2]	(1,2,4,6,8,10,12,14,8,3,8)	0	76	2
(8,1)	[1,0,0,0,0,0,0,2,0]	(1,2,4,6,8,10,12,14,8,4,8)	-2	77	3
(8,1)	[0,1,0,0,1,0,0,0,0]	(1,2,3,5,7,9,12,15,10,5,8)	-4	77	7
(8,1)	[1,0,0,0,0,0,1,0,1]	(1,2,4,6,8,10,12,14,9,4,8)	-4	78	11
(8,1)	[1,0,0,0,0,1,0,0,0]	(1,2,4,6,8,10,12,15,10,5,8)	-6	81	13
(8,1)	[0,0,0,0,0,0,0,0,3]	(1,3,5,7,9,11,13,15,9,3,8)	-2	84	3
(8,1)	[0,0,0,0,0,0,0,1,1]	(1,3,5,7,9,11,13,15,9,4,8)	-6	85	10
(8,1)	[0,0,0,0,0,0,1,0,0]	(1,3,5,7,9,11,13,15,10,5,8)	-8	87	10
(9,1)	[0,1,0,0,1,0,0,1,1]	(1,2,3,5,7,9,12,15,9,4,9)	2	76	1
(9,1)	[0,0,1,1,0,0,1,0,0]	(1,2,3,4,6,9,12,15,10,5,9)	2	76	1
(9,1)	[2,0,0,0,1,0,1,0,0]	(1,1,3,5,7,9,12,15,10,5,9)	2	77	1
(9,1)	[0,1,0,0,0,2,0,0,0]	(1,2,3,5,7,9,11,15,10,5,9)	2	77	1
(9,1)	[1,1,1,0,0,0,0,1,0]	(1,1,2,4,7,10,13,16,10,5,9)	2	78	1
(9,1)	[0,1,0,0,1,0,1,0,0]	(1,2,3,5,7,9,12,15,10,5,9)	0	78	2
(9,1)	[2,0,0,1,0,0,0,0,2]	(1,1,3,5,7,10,13,16,10,4,9)	2	79	1
(9,1)	[1,0,0,0,0,0,2,0,1]	(1,2,4,6,8,10,12,14,9,4,9)	2	79	1

JHEP02(2004)033

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(9,1)	[2,0,0,1,0,0,0,1,0]	(1,1,3,5,7,10,13,16,10,5,9)	0	80	2
(9,1)	[1,0,0,0,0,1,0,1,1]	(1,2,4,6,8,10,12,15,9,4,9)	0	80	3
(9,1)	[0,1,0,1,0,0,0,0,2]	(1,2,3,5,7,10,13,16,10,4,9)	0	80	3
(9,1)	[0,0,2,0,0,0,0,1,0]	(1,2,3,4,7,10,13,16,10,5,9)	0	80	2
(9,1)	[0,1,0,1,0,0,0,1,0]	(1,2,3,5,7,10,13,16,10,5,9)	-2	81	7
(9,1)	[1,0,0,0,0,1,1,0,0]	(1,2,4,6,8,10,12,15,10,5,9)	-2	82	6
(9,1)	[1,0,0,0,1,0,0,0,2]	(1,2,4,6,8,10,13,16,10,4,9)	-2	83	6
(9,1)	[1,2,0,0,0,0,0,0,1]	(1,1,2,5,8,11,14,17,11,5,9)	0	84	2
(9,1)	[1,0,0,0,1,0,0,1,0]	(1,2,4,6,8,10,13,16,10,5,9)	-4	84	13
(9,1)	[2,0,1,0,0,0,0,0,1]	(1,1,3,5,8,11,14,17,11,5,9)	-2	85	5
(9,1)	[0,0,0,0,0,0,1,0,3]	(1,3,5,7,9,11,13,15,9,3,9)	2	85	1
(9,1)	[0,1,1,0,0,0,0,0,1]	(1,2,3,5,8,11,14,17,11,5,9)	-4	86	12
(9,1)	[0,0,0,0,0,0,1,1,1]	(1,3,5,7,9,11,13,15,9,4,9)	-2	86	4
(9,1)	[1,0,0,1,0,0,0,0,1]	(1,2,4,6,8,11,14,17,11,5,9)	-6	88	24
(9,1)	[0,0,0,0,0,1,0,0,2]	(1,3,5,7,9,11,13,16,10,4,9)	-4	88	9
(9,1)	[0,0,0,0,0,0,2,0,0]	(1,3,5,7,9,11,13,15,10,5,9)	-4	88	4
(9,1)	[0,0,0,0,0,1,0,1,0]	(1,3,5,7,9,11,13,16,10,5,9)	-6	89	16
(9,1)	[2,1,0,0,0,0,0,0,0]	(1,1,3,6,9,12,15,18,12,6,9)	-4	92	4
(9,1)	[0,0,0,0,1,0,0,0,1]	(1,3,5,7,9,11,14,17,11,5,9)	-8	92	24
(9,1)	[0,2,0,0,0,0,0,0,0]	(1,2,3,6,9,12,15,18,12,6,9)	-6	93	6
(9,1)	[1,0,1,0,0,0,0,0,0]	(1,2,4,6,9,12,15,18,12,6,9)	-8	94	20
(9,1)	[0,0,0,1,0,0,0,0,0]	(1,3,5,7,9,12,15,18,12,6,9)	-10	97	20
(10,1)	[0,1,0,1,0,1,0,0,1]	(1,2,3,5,7,10,13,17,11,5,10)	2	84	1
(10,1)	[1,0,0,0,1,0,1,1,0]	(1,2,4,6,8,10,13,16,10,5,10)	2	85	1
(10,1)	[2,0,1,0,0,0,1,0,1]	(1,1,3,5,8,11,14,17,11,5,10)	2	86	1
(10,1)	[0,1,1,0,0,0,0,2,0]	(1,2,3,5,8,11,14,17,10,5,10)	2	86	1
(10,1)	[2,0,0,1,1,0,0,0,0]	(1,1,3,5,7,10,14,18,12,6,10)	2	87	1
(10,1)	[1,0,0,1,0,0,0,1,2]	(1,2,4,6,8,11,14,17,10,4,10)	2	87	1
(10,1)	[1,0,0,0,1,1,0,0,1]	(1,2,4,6,8,10,13,17,11,5,10)	0	87	3
(10,1)	[0,1,1,0,0,0,1,0,1]	(1,2,3,5,8,11,14,17,11,5,10)	0	87	3
(10,1)	[0,0,2,0,1,0,0,0,0]	(1,2,3,4,7,10,14,18,12,6,10)	2	87	1
(10,1)	[1,2,0,0,0,1,0,0,0]	(1,1,2,5,8,11,14,18,12,6,10)	2	88	1
(10,1)	[1,0,0,1,0,0,0,2,0]	(1,2,4,6,8,11,14,17,10,5,10)	0	88	2
(10,1)	[0,1,0,1,1,0,0,0,0]	(1,2,3,5,7,10,14,18,12,6,10)	0	88	2
(10,1)	[2,0,1,0,0,1,0,0,0]	(1,1,3,5,8,11,14,18,12,6,10)	0	89	2
(10,1)	[1,0,0,1,0,0,1,0,1]	(1,2,4,6,8,11,14,17,11,5,10)	-2	89	10
(10,1)	[0,0,0,0,0,1,1,0,2]	(1,3,5,7,9,11,13,16,10,4,10)	2	89	1
(10,1)	[0,1,1,0,0,1,0,0,0]	(1,2,3,5,8,11,14,18,12,6,10)	-2	90	7
(10,1)	[0,0,0,0,0,1,1,1,0]	(1,3,5,7,9,11,13,16,10,5,10)	0	90	2
(10,1)	[2,1,0,0,0,0,0,1,1]	(1,1,3,6,9,12,15,18,11,5,10)	0	91	3
(10,1)	[1,0,0,0,2,0,0,0,0]	(1,2,4,6,8,10,14,18,12,6,10)	-2	91	3

(n_c, n_0)	p_i	β	β^2	$ht(\beta)$	μ
(10,1)	[0,2,0,0,0,0,0,0,3]	(1,2,3,6,9,12,15,18,11,4,10)	2	91	1
(10,1)	[0,0,0,0,1,0,0,1,2]	(1,3,5,7,9,11,14,17,10,4,10)	0	91	2
(10,1)	[1,0,1,0,0,0,0,0,3]	(1,2,4,6,9,12,15,18,11,4,10)	0	92	2
(10,1)	[1,0,0,1,0,1,0,0,0]	(1,2,4,6,8,11,14,18,12,6,10)	-4	92	14
(10,1)	[0,2,0,0,0,0,0,1,1]	(1,2,3,6,9,12,15,18,11,5,10)	-2	92	7
(10,1)	[0,0,0,0,1,0,0,2,0]	(1,3,5,7,9,11,14,17,10,5,10)	-2	92	4
(10,1)	[0,0,0,0,0,2,0,0,1]	(1,3,5,7,9,11,13,17,11,5,10)	-2	92	5
(10,1)	[2,1,0,0,0,0,1,0,0]	(1,1,3,6,9,12,15,18,12,6,10)	-2	93	5
(10,1)	[1,0,1,0,0,0,0,1,1]	(1,2,4,6,9,12,15,18,11,5,10)	-4	93	19
(10,1)	[0,0,0,0,1,0,1,0,1]	(1,3,5,7,9,11,14,17,11,5,10)	-4	93	13
(10,1)	[0,2,0,0,0,0,1,0,0]	(1,2,3,6,9,12,15,18,12,6,10)	-4	94	9
(10,1)	[1,0,1,0,0,0,1,0,0]	(1,2,4,6,9,12,15,18,12,6,10)	-6	95	30
(10,1)	[0,0,0,1,0,0,0,0,3]	(1,3,5,7,9,12,15,18,11,4,10)	-2	95	5
(10,1)	[0,0,0,1,0,0,0,1,1]	(1,3,5,7,9,12,15,18,11,5,10)	-6	96	27
(10,1)	[0,0,0,0,1,1,0,0,0]	(1,3,5,7,9,11,14,18,12,6,10)	-6	96	15
(10,1)	[3,0,0,0,0,0,0,0,2]	(1,1,4,7,10,13,16,19,12,5,10)	-2	98	3
(10,1)	[0,0,0,1,0,0,1,0,0]	(1,3,5,7,9,12,15,18,12,6,10)	-8	98	34
(10,1)	[3,0,0,0,0,0,0,1,0]	(1,1,4,7,10,13,16,19,12,6,10)	-4	99	5
(10,1)	[1,1,0,0,0,0,0,0,2]	(1,2,4,7,10,13,16,19,12,5,10)	-6	99	24
(10,1)	[1,1,0,0,0,0,0,1,0]	(1,2,4,7,10,13,16,19,12,6,10)	-8	100	42
(10,1)	[0,0,1,0,0,0,0,0,2]	(1,3,5,7,10,13,16,19,12,5,10)	-8	101	31
(10,1)	[0,0,1,0,0,0,0,1,0]	(1,3,5,7,10,13,16,19,12,6,10)	-10	102	57
(10,1)	[2,0,0,0,0,0,0,0,1]	(1,2,5,8,11,14,17,20,13,6,10)	-10	107	32
(10,1)	[0,1,0,0,0,0,0,0,1]	(1,3,5,8,11,14,17,20,13,6,10)	-12	108	58
(10,1)	[1,0,0,0,0,0,0,0,0]	(1,3,6,9,12,15,18,21,14,7,10)	-14	116	27

References

[1] S. Ferrara, J. Scherk and B. Zumino, *Algebraic properties of extended supergravity theories*, *Nucl. Phys.* **B 121** (1977) 393;
E. Cremmer, J. Scherk and S. Ferrara, SU(4) *invariant supergravity theory*, *Phys. Lett.* **B 74** (1978) 61.

[2] E. Cremmer and B. Julia, *The $N = 8$ supergravity theory, 1. The lagrangian*, *Phys. Lett.* **B 80** (1978) 48.

[3] P.C. West, *Hidden superconformal symmetry in M-theory*, *J. High Energy Phys.* **08** (2000) 007 [hep-th/0005270].

[4] P.C. West, *E_{11} and M-theory*, *Class. and Quant. Grav.* **18** (2001) 4443 [hep-th/0104081].

[5] I. Schnakenburg and P.C. West, *Kac-moody symmetries of IIB supergravity*, *Phys. Lett.* **B 517** (2001) 421 [hep-th/0107181].

[6] T. Damour, M. Henneaux and H. Nicolai, *E_{10} and a 'small tension expansion' of M-theory*, *Phys. Rev. Lett.* **89** (2002) 221601 [hep-th/0207267].

JHEP02(2004)033

[7] T. Damour, M. Henneaux and H. Nicolai, *Cosmological billiards*, *Class. and Quant. Grav.* **20** (2003) R145 [hep-th/0212256].

[8] E. Cremmer, B. Julia, H. Lu and C.N. Pope, *Dualisation of dualities, I*, *Nucl. Phys.* **B 523** (1998) 73 [hep-th/9710119].

[9] P. West, E_{11}, SL(32) *and central charges*, *Phys. Lett.* **B 575** (2003) 333 [hep-th/0307098].

[10] F. Englert and L. Houart, $\mathcal{G}^{+++}$ *invariant formulation of gravity and M-theories: exact BPS solutions*, *J. High Energy Phys.* **01** (2004) 002 [hep-th/0311255].

[11] H. Nicolai and T. Fischbacher, *Low level representations for E_{10} and E_{11}*, hep-th/0301017.

[12] N.D. Lambert and P.C. West, *Coset symmetries in dimensionally reduced bosonic string theory*, *Nucl. Phys.* **B 615** (2001) 117 [hep-th/0107209].

[13] M.R. Gaberdiel, D.I. Olive and P.C. West, *A class of lorentzian Kac-Moody algebras*, *Nucl. Phys.* **B 645** (2002) 403 [hep-th/0205068].

[14] F. Englert, L. Houart, A. Taormina and P. West, *The symmetry of M-theories*, *J. High Energy Phys.* **09** (2003) 020 [hep-th/0304206].

[15] F. Englert, L. Houart and P. West, *Intersection rules, dynamics and symmetries*, *J. High Energy Phys.* **08** (2003) 025 [hep-th/0307024].

[16] A. Kleinschmidt, I. Schnakenburg and P. West, *Very-extended Kac-Moody algebras and their interpretation at low levels*, hep-th/0309198.

[17] P. West, *Very-extended E_8 and A_8 at low levels, gravity and supergravity*, *Class. and Quant. Grav.* **20** (2003) 2393, [hep-th/0212291].

[18] V. Kac, *Infinite dimensional algebras*, Cambridge University Press, Cambridge 1990.

[19] A. Kleinschmidt, *E_{11} as E_{10} representation at low levels*, *Nucl. Phys.* **B 677** (2004) 553 [hep-th/0304246].

[20] K.S. Stelle, *BPS branes in supergravity*, hep-th/9803116.

[21] J.A. de Azcarraga, J.P. Gauntlett, J.M. Izquierdo and P.K. Townsend, *Topological extensions of the supersymmetry algebra for extended objects*, *Phys. Rev. Lett.* **63** (1989) 2443.

JHEP02(2004)033